Research Express

On the Small Dispersion Limit for a Special Class of Complex Ginzburg-Landau Equations

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0 Introduction

The Ginzburg-Landau type equations are simplified mathematical models for non-linear systems in mechanics, physics, and other areas. The time-dependent complex Ginzburg-Landau partial differential equation has been used to model phenomena in a number of different areas in physics, including phase transitions in non-equilibrium systems, instabilities in hydrodynamic systems, chemical turbulence, and thermodynamics([1]).

There have been many discussions on the Ginzburg-Landau equations([2,3]). C. D. Levermore and M. Oliver [2] studied the following Ginzburg-Landau equations:

$$u_t = Ru + (1 + i\nu)\Delta u - (1 + i\mu)|u|^{2\sigma}u,$$

here $i = \sqrt{-1}$, R, ν and μ are given real numbers. $R > 0$ is the instability parameter and ν is the dispersive parameter of either sign. In [2], the authors proved the existence and uniqueness of local (in time) classical solutions-even C^∞ solutions for σ is a positive integer, and initial data in the classes L^p and H^q. Then they established global classical solutions through a priori bounds of L^p norms and does not restrict σ to integer values. D. R. Jefferson[3] considered the following class of complex Ginzburg-Landau equations:

$$v_t = \delta_1\Delta v + i|v|^{2\sigma}v, \quad \sigma \in \mathbf{N}, t > 0, \tag{1}'$$

with initial value condition

$$v(0,x) = u_0(x) \tag{2}'$$

and

$$\psi_t = i|\psi|^{2\sigma}\psi, \quad \sigma \in \mathbf{N}, t > 0, \tag{1}''$$

with initial value condition

$$\psi(0,x) = u_0(x). \tag{2}''$$

The authors firstly proved for C^m-smooth initial data, there was a unique global C^m-smooth classical solution to problem (1)′, (2)′. Then they proved for C^{m+2}-smooth initial data u_0, the solution to problem (1)′,(2)′ was approximated by the corresponding dissipation-free equation (1), (2) for time of order $\ln(\delta_1^{-1})$ in the C^m-norm.

In this paper, we consider the complex Ginzburg-Landau equations

$$u_t = (\delta_1 + i\delta_2)\triangle u + i\beta|u|^{2\sigma}u, \quad t > 0, \tag{1}$$

with initial value condition

$$u(0,x) = u_0(x), \tag{2}$$

here $i = \sqrt{-1}$, $\sigma \in \mathbf{N}$, $\delta_1 > 0$, δ_2, β are real numbers. We study the periodic solutions in d spatial dimensions, so

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our spatial domain is the d-dimensional torus T^d. In section 2 we show the existence and uniqueness of global (in time) solution for the problem (1), (2) with the initial condition belonging to $W^{1,p}(T^d)$. In section 3 we show that the solution to problem (1), (2) is approximated in all $W^{1,p}(T^d)$-norm by solutions of the corresponding small dispersion limit equation for a period of time that goes to infinity as δ_2 goes to zero. Due to the dispersive term $i\delta_2\triangle u$ and lack of the dissipative term $-|u|^{2\sigma}u$, we use different method with C. D. Levermore, M. Oliver[2] and D. R. Jefferson[3] to prove the existence of the global solution to problem (1), (2).

1 Existence of the Solution

In addition, throughout this paper, we set $W^{1,p}=W^{1,p}(T^d)$, $L^p=L^p(T^d)$ to denote the usual Sobolev spaces. The following lemma gives the existence of a unique mild solution to the problem (1), (2) on a finite time interval.

Lemma 1 (Local Existence). Let $u_0\in W^{1,p}, p>d$. Then there exists a $T>0$ such that the problem (1), (2) has a unique mild solution in the space

$$S=\{w\in C([0,T],W^{1,p})\mid \sup_{t\in[0,T]}\|w\|_{W^{1,p}}\leqslant M\|u_0\|_{W^{1,p}}+1=R\}.$$

In order to show that the solution exists for all $t>0$, we only need some conditions such that

$$\|u\|_{W^{1,p}}<\infty,\ \text{for all } t>0. \tag{3}$$

This can be achieved by the following a priori estimates, that is

$$\|u\|_{W^{1,p}}\leqslant C(T,\|u_0\|_{W^{1,p}}),\ t\in[0,T],\ T>0,$$

this mean $\|u\|_{W^{1,p}}$ cannot go to infinity at finite time. In order to establish (3), we derive a priori estimates for the solution of problem (1), (2) in the following lemmas.

Lemma 2 Assume p, δ_1, δ_2 satisfy

$$2\leqslant p<\frac{2\sqrt{\delta_1^2+\delta_2^2}}{\sqrt{\delta_1^2+\delta_2^2}-\delta_1}\quad\text{or}\quad\frac{2\sqrt{\delta_1^2+\delta_2^2}}{\delta_1+\sqrt{\delta_1^2+\delta_2^2}}<p\leqslant 2, \tag{4}$$

if $u_0\in W^{1,p}$ and $p>\sigma d$, then for any $t\in[0,T]$, we have

$$\|u\|_{W^{1,p}}\leqslant C(T,\|u_0\|_{W^{1,p}})$$

A priori bounds given by lemma 2, together with lemma 1 imply for $u_0\in W^{1,p}$ initial data, we have a unique global solution $u\in C([0,\infty),W^{1,p})$, and according to the regularity of parabolic equations, we finally get the following global existence result.

Theorem 1 (Global Existence) Assume p, δ_1, δ_2 satisfy (4), $p>\sigma d$. Then for any $u_0\in W^{1,p}$, the problem (1), (2) possess a unique solution. This solution is in $C([0,\infty),W^{1,p})\cap C^\infty((0,\infty)\times T^d)$.

2 Appoximation of Solution

We now consider that for $W^{2,p}$ initial data u_0, the solution to problem (1), (2) is approximated in the $W^{1,p}$-norm by the solution to the following problem:

$$v_t=\delta_1\Delta v+i\beta|v|^{2\sigma}v,\quad \sigma\in\mathbf{N}, t>0, \tag{5}$$

with initial value condition

$$v(0,x)=u_0(x). \tag{6}$$

Lemma 3 Assume $p>d$, if u is the solution to problem (1), (2) with $u_0\in W^{1,p}$, and v is the solution to problem (5), (6), then

$$\|u-v\|_{L^p}^p\leqslant\frac{K_1|\delta_2|}{|\delta_2|+1}e^{K_2(|\delta_2|+1)t}.$$

Theorem 2 Assume $p>\max\{\sigma d,2\}$ and $\delta_1\geqslant\frac{P-1}{4}|\delta_2|$, if u is the solution to problem (1), (2) with $u_0\in W^{2,p}$, and v is the solution to problem (5), (6), then

$$\| u - v \|_{W^{1,p}} \leqslant \left(\frac{K_3 \mid \delta_2 \mid}{\mid \delta_2 \mid + 1} \right)^{\frac{1}{p} e^{K_4 (\mid \delta_2 \mid + 1) t}},$$

where K_3 and K_4 are positive constant which depend $\| u_0 \|_{W^{1,p}}, p, \beta, \sigma$. In particular, for any $\varepsilon > 0$ we have that

$$\| u - v \|_{W^{1,p}} \leqslant \varepsilon \text{ for } 0 \leqslant t \leqslant \frac{1}{K_4 (\mid \delta_2 \mid + 1)} \left| \ln \left[\varepsilon \left(\frac{K_3 \mid \delta_2 \mid}{\mid \delta_2 \mid + 1} \right)^{-\frac{1}{p}} \right] \right|.$$

In [3], the author proved the following theorem.

Theorem A Let m be any nonnegative integer. If v is the solution to (1)′, (2)′ with $u_0 \in C^{m+2}(T^d)$, and ψ is the solution to (1)″, (2)″, then

$$\| v - \psi \|_{C^m} \leqslant \delta_1 \mathscr{A} e^{Kt},$$

where $\mathscr{A}$ and K are positive constants which depend on $\| u_0 \|_{C^{m+2}}, p, m$ and d. In particular, for any $\varepsilon > 0$ we have

$$\| v - \psi \|_{C^m} \leqslant \varepsilon \text{ for } 0 \leqslant t \leqslant \frac{1}{K} \mid \ln \left(\frac{\varepsilon}{\delta_1 \mathscr{A}} \right) \mid.$$

In the following, we consider that for $C^2 \cap W^{1,p}$ initial data u_0, the solution to problem (1), (2) is approximated in the C^0-norm by the solution to the following problem:

$$\psi_t = \mathrm{i}\beta \mid \psi \mid^{2\sigma} \psi, \quad \sigma \in \mathbf{N}, t > 0, \tag{7}$$

with initial value condition

$$\psi(0, x) = u_0(x). \tag{8}$$

By the results of [3], theorem 2 and $W_0^{1,p}(T^d) \hookrightarrow C^0(T^d)$ for $p > d$, we get

Theorem 3 Assume $p > \max\{d, 2\}$ and $\delta_1 \geqslant \frac{P-1}{4} \mid \delta_2 \mid$, if u is the solution to problem (1), (2) with $u_0 \in C^2 \cap W^{2,p}$, and v is the solution to problem (5), (6), ψ is the solution to problem (7), (8), then

$$\| u - \psi \|_{C^0} \leqslant \varepsilon \text{ for } 0 \leqslant t \leqslant \min \left(\frac{1}{K} \mid \ln \left(\frac{\varepsilon}{\delta_1 \mathscr{A}} \right) \mid, \frac{1}{K_4 (\mid \delta_2 \mid + 1)} \mid \ln \left[\varepsilon \left(\frac{K_3 \mid \delta_2 \mid}{\mid \delta_2 \mid + 1} \right)^{-\frac{1}{p}} \right] \mid \right),$$

where $\mathscr{A}$ and K are positive constants which depend on $\| u_0 \|_{C^2}, p, m, \beta$, and d, K_3, K_4 are positive constants which depend on $\| u_0 \|_{W^{2,p}}, p, \beta, \sigma$.

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