A Generalization of Ekeland Variational Principle and Its Applications

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Abstract: In this paper, classical Ekeland variational principle is generalized. The theoretical result is applied to optimization problems with regular constraints.

Key words: variational principle, general principle on ordered set, approximately minimal point, optimization

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0 Introduction

In 1974, I. Ekeland[1] put forward an variational principle:

Theorem A (Ekeland variational principle). Let X be a complete metric space, $\phi: X \to R \cup \{+\infty\}$, $\not\equiv +\infty$, be bounded from below and lower semi-continuous. If there exist $\varepsilon > 0$ and $x \in X$ such that

$$\phi(x) \leqslant \inf_X \phi + \varepsilon,$$

then there exists $y \in X$ such that

$$\phi(y) \leqslant \phi(x);$$
$$d(x,y) \leqslant 1;$$
$$\phi(z) > \phi(y) - \varepsilon d(y,z), \ \forall z \neq y.$$

This variational principle had been applied to many fields, including control theory, optimal theory, geometry in Banach spaces and big area analysis, etc[2][3].

In paper [6], S. Park generalized Ekeland's variational principle to quasi complete metric space, and he still suppose that the function $\phi(x)$ is bounded from below on X. In this paper, we weaken the condition that $\phi(x)$ is bounded from below on X essentially, which makes the area of suited functional enlarged.

We know that the classical general principle on ordered sets[4] can deduce Theorem A. In this paper, we utilize the generalized general principle on ordered sets[5] to prove the main result in this paper.

Lemma 1 (generalized general principle on ordered sets) [5]. Assume that X is a partial ordered set and a Hausdorff topological space, which satisfies

(i) $\forall x \in X$, $\{y \in X \mid y \geqslant x\}$ is a sequential closed set;

(ii) if $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots$, then $\{x_n\}$ has a convergent subsequence;

(iii) there exists $\psi: X \to R$ such that

$$x \in X,\ y \in X,\ x \leqslant y,\ x \neq y, \Rightarrow \psi(x) < \psi(y).$$

Then X has maximal element.

1 Main Results

Theorem 1 Let X be a complete metric space. Suppose $\phi: X \to R \cup \{+\infty\}$, $\not\equiv +\infty$ is lower semi-continuous and bounded from below on each bounded set. If there exists $x_0 \in X$ such that

$$\liminf_{d(x,x_0) \to +\infty} \frac{\phi(x)}{d(x,x_0)} \geqslant 0,$$

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then $\forall \varepsilon > 0$, ϕ has ε-approximately minimal point, i.e., $\exists x^* \in X$ such that

$$\phi(x) > \phi(x^*) - \varepsilon d(x, x^*), \ \forall x \neq x^*.$$

Proof From $\liminf\limits_{d(x,x_0)\to+\infty} \dfrac{\phi(x)}{d(x,x_0)} \geqslant 0$, we have for all $\varepsilon > 0$, there exists $M > 0$ such that $\phi(x) > -\frac{\varepsilon}{2} d(x,x_0)$ when $d(x,x_0) \geqslant M$. In addition, ϕ is bounded from below on each bounded set, then there exists $\beta > 0$ such that $\phi(x) > -\beta$ when $d(x,x_0) \leqslant M$. Hence $\phi(x) > -\frac{\varepsilon}{2} d(x,x_0) - \beta$, $\forall x \in X$.

For $x, y \in X$, define $x \leqslant y$ provided

$$\frac{\varepsilon}{2} d(x,y) \leqslant \phi(x) + \frac{\varepsilon}{2} d(x,x_0) - \left(\phi(y) + \frac{\varepsilon}{2} d(y,x_0)\right).$$

Step one: to prove that " $\leqslant$ " defined above satisfies the three axioms of partial order.

(i) $x \leqslant x$ is obvious; (ii) if $x \leqslant y$, $y \leqslant z$, then

$$\frac{\varepsilon}{2} d(x,y) \leqslant \phi(x) + \frac{\varepsilon}{2} d(x,x_0) - \left[\phi(y) + \frac{\varepsilon}{2} d(y,x_0)\right];$$

and

$$\frac{\varepsilon}{2} d(y,z) \leqslant \phi(y) + \frac{\varepsilon}{2} d(y,x_0) - \left[\phi(z) + \frac{\varepsilon}{2} d(z,x_0)\right].$$

From the above two inequations, we have

$$\frac{\varepsilon}{2} d(x,z) \leqslant \frac{\varepsilon}{2} d(x,y) + \frac{\varepsilon}{2} d(y,z) \leqslant \phi(f(x)) + \frac{\varepsilon}{2} d(x,x_0) - \left[\phi(f(z)) + \frac{\varepsilon}{2} d(z,x_0)\right],$$

i.e., $x \leqslant z$; (iii) if $x \leqslant y$, $y \leqslant x$, then as the process of (ii), we have

$$\frac{\varepsilon}{2} d(x,y) + \frac{\varepsilon}{2} d(y,x) \leqslant 0,$$

which yields $x = y$.

Step two: to prove that there exists a functional $\psi: X \to R$ such that

$$x \in X, \ y \in X, \ x \leqslant y, \ x \neq y \Rightarrow \psi(x) < \psi(y).$$

Let $x \leqslant y$, $x \neq y$, then

$$0 < \frac{\varepsilon}{2} d(x,y) \leqslant \phi(x) + \frac{\varepsilon}{2} d(x,x_0) - \left(\phi(y) + \frac{\varepsilon}{2} d(y,x_0)\right),$$

i.e.,

$$\phi(x) + \frac{\varepsilon}{2} d(x,x_0) > \phi(y) + \frac{\varepsilon}{2} d(y,x_0).$$

Let $\psi: X \to R$ be $\psi(x) = -\phi(x) - \frac{\varepsilon}{2} d(x,x_0)$, then $\psi(x) < \psi(y)$, i.e., ψ is strictly decreasing under the defined above partial order.

Step three: to prove that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \cdots \Rightarrow \{x_n\}$ is convergent.

Let $p(x) = \phi(x) + \frac{\varepsilon}{2} d(x,x_0)$, then

$$0 \leqslant \frac{\varepsilon}{2} d(x_n, x_m) \leqslant p(x_n) - p(x_m), \ \forall m > n.$$

Hence $\{p(x_n)\}$ is monotone decreasing. On the other hand, $p(x) \geqslant -\beta$, $\forall x \in X$, so $\{p(x_n)\}$ is a Cauchy sequence, hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ is convergent.

Step four: to prove that $\forall x \in X$, $\{y \in X \mid y \geqslant x\}$ is sequentially closed.

Assume that $y_n \geqslant x$, and $y_n \to y$, then

$$\frac{\varepsilon}{2} d(x,y_n) \leqslant \phi(x) + \frac{\varepsilon}{2} d(x,x_0) - \left(\phi(y_n) + \frac{\varepsilon}{2} d(y_n,x_0)\right).$$

Since ϕ is lower semi-continuous, we have $\phi(y) \leqslant \liminf\limits_{n\to\infty} \phi(y_n)$, i.e., $\limsup\limits_{n\to\infty}(-\phi(y_n)) \leqslant -\phi(y)$. Taking limits with respect to n yields

$$\frac{\varepsilon}{2}d(x,y)=\frac{\varepsilon}{2}\lim_{n\to\infty}d(x,y_n)\leqslant\phi(x)+\frac{\varepsilon}{2}d(x,x_0)+\lim_{n\to\infty}\sup(-\phi(y_n))-\frac{\varepsilon}{2}d(y,x_0)$$

$$\leqslant\phi(x)+\frac{\varepsilon}{2}d(x,x_0)-(\phi(y)+\frac{\varepsilon}{2}d(y,x_0)),$$

i.e., $x\leqslant y$. Hence $\{y\in X\mid y\geqslant x\}$ is a sequentially closed set.

From the above four steps, all the conditions of Lemma 1 are satisfied. Hence X has maximal element under the partial order, i.e., $\exists x^*\in X$ such that $\forall x\neq x^*$, $x^*\not\leqslant x$, which means

$$\frac{\varepsilon}{2}d(x^*,x)>\phi(x^*)+\frac{\varepsilon}{2}d(x^*,x_0)-(\phi(x)+\frac{\varepsilon}{2}d(x,x_0)),$$

therefore

$$\phi(x)>\phi(x^*)+\frac{\varepsilon}{2}d(x^*,x_0)+\alpha d(x,x_0)-\frac{\varepsilon}{2}d(x^*,x)$$

$$\geqslant\phi(x^*)-\frac{\varepsilon}{2}d(x^*,x)-\frac{\varepsilon}{2}d(x^*,x)$$

$$=\phi(x^*)-\varepsilon d(x^*,x).$$

Remark 1 The inequation in Theorem 1 has nothing with the selection of x_0.

Remark 2 Theorem 1 generalizes classical Ekeland variational principle. For example, $f(x)=x^{\frac{1}{3}}$, $x\in\mathbf{R}$, satisfies all the conditions of Theorem 1, however we can 't get a ε-approximation minimal point of $f(x)$ from classical Ekeland variational principle.

Remark 3 Theorem 1 generalizes [6, Theorem 3]. Indeed, from the proof of Theorem 1, we can see that it 's not needed that the symmetry of the metric d, so Theorem 1 can be generalized to quasi metric space.

Corollary 1 Assume that X is a reflexive Banach space, $\phi:X\to\mathbf{R}\cup\{+\infty\}$, $\not\equiv+\infty$, is weakly lower semi-continuous and satisfies

$$\lim_{\|x\|\to\infty}\inf\frac{\phi(x)}{\|x\|}\geqslant 0.$$

Then $\forall\varepsilon>0$, ϕ has ε-approximately minimal point.

Proof Since X is reflexive, we have $\forall M>0$, $B=\{x\in X\mid\|x\|\leqslant M\}$ is weakly sequential compact. Note that B is convex closed, then B is a weakly closed set. Since ϕ is weakly lower-continuous, we know that ϕ is bounded from below on the weakly sequential compact and weakly closed set B. The conclusion is proved by Theorem 1.

Corollary 2 Assume that X is a reflexive Banach space, $\phi:X\to\mathbf{R}\cup\{+\infty\}$, $\not\equiv+\infty$, is a convex lower semi-continuous functional, and satisfies

$$\lim_{\|x\|\to\infty}\inf\frac{\phi(x)}{\|x\|}\geqslant 0.$$

Then $\forall\varepsilon>0$, ϕ has ε-approximately minimal point.

Proof From [8], ϕ being convex and lower semi-continuous yields that ϕ is weakly lower semi-continuous on the weakly convex set $\{x\in X\mid\|x\|\leqslant M\}$, $\forall M>0$. The conclusion is proved by Corollary 1.

Corollary 3 Assume that $\phi:\mathbf{R}^N\to\mathbf{R}$ is lower semi-continuous, and there exists $x_0\in X$ such that

$$\lim_{|x-x_0|\to\infty}\inf\frac{\phi(x)}{|x-x_0|}\geqslant 0.$$

Then $\forall\varepsilon>0$, ϕ has ε-approximately minimal point.

Theorem 2 Let X be a Banach space and $\phi:X\to\mathbf{R}\cup\{+\infty\}$, $\not\equiv+\infty$ be Gâteaux derivative. Suppose ϕ is bounded from below on each bounded set and satisfies

$$\lim_{\|x\|\to\infty}\inf\frac{\phi(x)}{\|x\|}\geqslant 0.$$

Then $\forall\varepsilon>0$, $\exists x_\varepsilon\in X$ such that

(a) $\phi(x_\varepsilon)\leqslant\phi(x)+\varepsilon\|x-x_\varepsilon\|$, $\forall x\in X$.

(b) $\|\phi'(x_\varepsilon)\|\leqslant\varepsilon$.

Proof From Theorem 1, (a) is valid. Hence $\forall y \in X$,

$$\phi(x_\varepsilon + y) \geqslant \phi(x_\varepsilon) - \varepsilon \| y \| .$$

Since ϕ is Gâteaux derivative, there exists $\phi'(x_\varepsilon) \in X^*$ such that

$$\phi(x_\varepsilon + tu) - \phi(x_\varepsilon) = \phi'(x_\varepsilon)(tu) + o(\| tu \|), \ \forall u \in X.$$

Select u satisfying $\| u \| = 1$, then

$$\phi'(x_\varepsilon)(tu) + o(t) = \phi(x_\varepsilon + tu) - \phi(x_\varepsilon) \geqslant -\varepsilon \| tu \| = -\varepsilon |t| .$$

When $t > 0$, we have

$$\phi'(x_\varepsilon)(u) + \frac{o(t)}{t} \geqslant -\varepsilon,$$

i.e., $\phi'(x_\varepsilon)(u) \geqslant -\varepsilon$; When $t < 0$, we have

$$\phi'(x_\varepsilon)(u) + \frac{o(t)}{t} \leqslant \varepsilon,$$

i.e., $\phi'(x_\varepsilon)(u) \leqslant \varepsilon$. From above inequations, we have $|\phi'(x_\varepsilon)(u)| \leqslant \varepsilon$ for $u \in X$ satisfying $\| u \| = 1$, then $\| \phi'(x_\varepsilon) \| \leqslant \varepsilon$, i.e., (b) holds.

2 Optimization Problems with Regular Constraints

Let X be a Banach space, $F: X \to \mathbf{R}$ a Fréchet-differentiable function, $G_i: X \to \mathbf{R} (1 \leqslant i \leqslant m)$ m continuously Fréchet-differentiable functions. We consider the constrained optimization problem

$$\begin{gathered} \lim \inf F(x), \\ G_i(x) = 0, \ 1 \leqslant i \leqslant p, \\ G_i(x) \geqslant 0, \ p \leqslant i \leqslant m. \end{gathered} \tag{1}$$

We denote by C the feasible set

$$C = \{x \in X \mid G_i(x) = 0, 1 \leqslant i \leqslant p, G_i(x) \geqslant 0, p \leqslant i \leqslant m\}. \tag{2}$$

and by $I(x)$ the set of saturated constraints at a feasible point $x \in C$,

$$i \in I(x) \Leftrightarrow G_i(x) = 0. \tag{3}$$

We can now state our regularity assumption, which is of quite a standard type:

$$\forall x \in C, \{G_i(x) \mid i \in I(x)\} \text{ are linearly independent.} \tag{4}$$

It is clear that problem (1) is highly nonlinear, and it is difficult to get the solution in Banach space. Nevertheless, we can find points which are "almost" optimal and which "almost" satisfy the necessary conditions for optimality. First we give two lemmas:

Lemma 2[1] Let $\varepsilon > 0$, there exists $x_\varepsilon \in X$, for every $h \in X$ such that

$$\langle G'_i(x_\varepsilon), h \rangle = 0, \ \forall i \in \{1, \cdots, p\};$$
$$\langle G'_i(x_\varepsilon), h \rangle \geqslant 0, \ \forall i \in \{p+1, \cdots, m\} \cap I(x_\varepsilon).$$

Then $\langle F'(x_\varepsilon), h \rangle \geqslant -\varepsilon \| h \|$.

Lemma 3[1] Let $u_i^* (1 \leqslant i \leqslant p)$, $v_j^* (1 \leqslant j \leqslant q)$ and w^* be linear functionals on X such that

$$\left.\begin{array}{l} \langle u_i^*, h \rangle = 0, \ 1 \leqslant i \leqslant p \\ \langle v_j^*, h \rangle \geqslant 0, \ 1 \leqslant j \leqslant q \end{array}\right. \Rightarrow \langle w^*, h \rangle \geqslant -\varepsilon \| h \|.$$

Then there exist p real number λ_i, $1 \leqslant i \leqslant p$ and q nonnegative number μ_j, $1 \leqslant j \leqslant q$ such that

$$\left\| w^* - \sum_{i=1}^{p} \lambda_i u_i^* - \sum_{j=1}^{q} \mu_j v_j^* \right\| < \varepsilon.$$

Theorem 3 Suppose F is Fréchet-differentiable and $G_i (1 \leqslant i \leqslant m)$ are C^1 functions satisfying assumption (4). Suppose moreover F is bounded from below on every bounded set and there exists $x_0 \in X$ such that

$$\liminf_{\| x \| \to \infty} \frac{F(x)}{\| x - x_0 \|} \geqslant 0.$$

Then for every $\varepsilon > 0$, there exists some point $x_\varepsilon \in C$ such that

$$F(x) \geqslant F(x_\varepsilon) - \varepsilon \| x - x_\varepsilon \|, \ \forall x \in C, \tag{5}$$

and there exist real numbers $\lambda_1, \cdots, \lambda_m$ such that

$$\left\| F'(x_\varepsilon) - \sum_{i=1}^{m} \lambda_i G'_i(x_\varepsilon) \right\| \leqslant \varepsilon, \tag{6}$$

where $\lambda_i \geqslant 0$, $\forall i \in \{p+1, \cdots, m\}$; If $G_i(x_\varepsilon) \neq 0$, then $\lambda_i = 0$.

Proof Step 1: to prove (5). Define a function $\bar{F}$ by

$$\begin{aligned} \bar{F}(x) &= +\infty, \ x \notin C, \\ \bar{F}(x) &= F(x), \ x \in C. \end{aligned} \tag{7}$$

Since G_i, $1 \leqslant i \leqslant m$ are continuous, we know that C is closed, so $\bar{F}$ is lower semi-continuous. It is obvious that

$$\lim_{\|x\| \to \infty} \inf \frac{\bar{F}(x)}{\| x - x_0 \|} \geqslant 0,$$

and $\bar{F}$ is bounded from below on every bounded set. Applying Theorem 1, we get a point $x_\varepsilon \in X$ such that

$$\bar{F}(x) > \bar{F}(x_\varepsilon) - \varepsilon \| x - x_\varepsilon \|, \ \forall x \neq x_\varepsilon.$$

From (7), we have $x_\varepsilon \in C$, hence

$$F(x) \geqslant F(x_\varepsilon) - \varepsilon \| x - x_\varepsilon \|, \ \forall x \in C.$$

Step 2: to prove (6). Since $G_i (1 \leqslant i \leqslant m)$ are Fréchet-differentiable, they are Gâteaux-differentiable, so for every $h \in X$, we have

$$\langle G'_i(x_\varepsilon), h \rangle = \lim_{t \to 0} \frac{G_i(x_\varepsilon + th) - G_i(x_\varepsilon)}{t}.$$

Since G_i is Fréchet-differentiable, we may restrict h such that $x_\varepsilon + th \in C$, then

$$\langle G'_i(x_\varepsilon), h \rangle = \begin{cases} \lim\limits_{t \to 0} \dfrac{0-0}{t} = 0, & i \in \{1, \cdots, p\}, \\ \lim\limits_{t \to 0} \dfrac{G_i(x_\varepsilon + th) - 0}{t} \geqslant 0, & i \in \{p+1, \cdots, m\} \cap I(x_\varepsilon). \end{cases}$$

From Lemma 2 and Lemma 3, Theorem 3 is proved.

Remark 4 Theorem 3 generalizes [1, Theorem 3.1].

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Ekeland 变分原理及其应用

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[摘要] 本文将 Ekeland 变分原理作了推广,并将理论结果应用到正则约束最优化问题上.

[关键词] 变分原理,序集一般原理,渐近极小点,最优化

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