

Coloring the Square of Outerplanar Graphs

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Abstract The square of a graph G , denoted by G^2 , is a graph with the same vertex set such that two vertices are adjacent in G^2 iff their distance is at most 2 in G . In this article we determine the chromatic number of the square of cycles. For outerplanar graphs we get the following main result: Let G be a connected simple outerplanar graph with maximum degree $\Delta(G)$, $G \neq C_5$. Then $\chi(G^2) \leq \Delta(G) + 2$.

Key words square, chromatic number, outerplanar graph

CLC number O157.5, **Document code** A, **Article ID** 1001-4616(2004)04-0028-04

0 Introduction

We assume that the given graph G is a connected simple finite graph. The set of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. A graph G is said to be an outerplanar graph if it can be drawn in the plane in such a way that all vertices are on the boundary of a same face. The length of a path between two vertices is the number of edges on that path. We define the distance between two vertices u and v , denoted by $\text{dist}_G(u, v)$, to be the length of the shortest path between them. The square of a graph G , denoted by G^2 , is a graph with the same vertex set such that two vertices are adjacent in G^2 iff their distance in G is at most 2. The degree of a vertex v is the number of edges incident with v , denoted by $d_G(v)$ or simply $d(v)$, if here is not confusing. If the degree of v is i , at least i or at most i we call it an i -vertex, a $\geq i$ -vertex or a $\leq i$ -vertex, respectively. $N_G[v]$ and $N_G(v)$ denote the neighbors of v in G including v and excluding v , respectively. Given an edge uv of a graph G , the graph G/uv is a new graph obtained from G by contracting the edge uv .

A vertex k -coloring of a graph G is a mapping $c: V(G) \rightarrow \{1, 2, \dots, k\}$ such that any two adjacent vertices u and v are mapped to different integers. The minimum k for which a k -coloring exists is called the chromatic number of G and is denoted by $\chi(G)$. The question of finding the best possible upper bound for the chromatic number of the square of a planar graph has been brought up by G. Wegner [1] first in 1977. He proposed the following conjecture:

Conjecture 0.1 (G. Wegner [1]) For a planar graph G :

$$\chi(G^2) \leq \begin{cases} \Delta + 5 & \text{if } 4 \leq \Delta \leq 7; \\ \lceil \frac{3}{2}\Delta + 1 \rceil & \text{if } \Delta \geq 8. \end{cases}$$

G. Wegner also gave examples illustrating that the bounds are best possible and showed that if $\Delta = 3$ then the square of G can be 8-colored. He conjectured that in fact 7 colors would be enough. Recently, Thomassen [2] has solved this conjecture for $\Delta = 3$. V. D. Heuvel and M. Guinness [3] showed that if G is a planar graph with maximum degree Δ , then $\chi(G^2) \leq 2\Delta + 25$. G. Agnarsson and M. M. Halldórsson [4] have a better asymptotic bound for large values of Δ . They proved that if G is a planar graph with $\Delta \geq 749$, then $\chi(G^2) \leq \lfloor \frac{9}{5}\Delta \rfloor + 2$. The best known bound for $\chi(G^2)$ for general planar graph is due to M. Molloy and M. R. Salavatipour [5]:

Theorem 0.1 For a planar graph G , $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$.

Instead of considering planar graphs , we consider outerplanar graphs and get the following main theorem :

Theorem 0.2 Let G be a connected outerplanar graph with $\Delta(G)$, $G \neq C_5$. Then $\chi(G^2) \leq \Delta(G) + 2$.

1 Preliminaries

Lemma 1.1 (Z. Zhang , J. Zhang and Wang [6]) Let G be a 2-connected simple outerplanar graph. Then there are at least two 2-vertices in G .

Theorem 1.1 Let $G = C_n$ where $n \geq 3$ and $n \neq 5$. If $n \equiv 0 \pmod{3}$ then $\chi(G^2) = 3 = \Delta(G) + 1$; if $n \equiv 1$ or $2 \pmod{3}$ then $\chi(G^2) = 4 = \Delta(G) + 2$.

Proof Let C_n be a cycle with vertices v_1, v_2, \dots, v_n . First, we assume that $n \equiv 0 \pmod{3}$. We define a mapping from $V(C_n)$ to the color set $\{1, 2, 3\}$ as follows :

$$\alpha(v_i) = \begin{cases} 3 & \text{if } i \equiv 0 \pmod{3} \\ 1 & \text{if } i \equiv 1 \pmod{3} \\ 2 & \text{if } i \equiv 2 \pmod{3} \end{cases} \quad \forall v_i \in V(C_n)$$

Thus, we get a 3-coloring c of C_n^2 . Obviously, $\chi(G^2) = 3$.

If $n \equiv 1$ or $2 \pmod{3}$, we can see that C_n^2 cannot be colored with 3 colors and we will prove $\chi(G^2) = 4$ in the following four cases :

Case 1 If $n \equiv 0 \pmod{4}$, we can assign color set $\{1, 2, 3, 4\}$ to the vertices in the following way :

$$\alpha(v_i) = \begin{cases} 4 & \text{if } i \equiv 0 \pmod{4} \\ 1 & \text{if } i \equiv 1 \pmod{4} \\ 2 & \text{if } i \equiv 2 \pmod{4} \\ 3 & \text{if } i \equiv 3 \pmod{4} \end{cases} \quad \forall v_i \in V(C_n)$$

Case 2 If $n \equiv 1 \pmod{4}$, $\forall v_i \in V(C_n) \setminus \{v_{n-4}, v_{n-3}, \dots, v_n\}$, we define $\alpha(v_i)$ the same as those in Case 1. Let $\alpha(v_{n-4}) = 1, \alpha(v_{n-3}) = 3, \alpha(v_{n-2}) = 4, \alpha(v_{n-1}) = 2$ and $\alpha(v_n) = 3$, respectively. Thus, we get a 4-coloring of C_n^2 .

Case 3 If $n \equiv 2 \pmod{4}$, $\forall v_i \in V(C_n) \setminus \{v_{n-1}, v_n\}$, we define $\alpha(v_i)$ the same as those in Case 1. Let $\alpha(v_{n-1}) = 2$ and $\alpha(v_n) = 3$. Obviously, c is a 4-coloring of C_n^2 .

Case 4 If $n \equiv 3 \pmod{4}$, similarly $\forall v_i \in V(C_n) \setminus \{v_{n-2}, v_{n-1}, v_n\}$, we can define $\alpha(v_i)$ the same as those in Case 1. Let $\alpha(v_{n-2}) = 2, \alpha(v_{n-1}) = 3$ and $\alpha(v_n) = 4$, respectively. We can check that c is a 4-coloring of C_n^2 .

Remark It is easy to see that $\chi(C_5^2) = 5$. For cycles and outerplanar graphs, it's the only case such that $\chi(G^2) = \Delta(G) + 3$.

Lemma 1.2 Let G be a 2-connected outerplanar graph with $\Delta(G) = 3$. Then there must be a 2-vertex v in G such that the number of vertices at distance one or two from v is at most 4.

Proof By contradiction. Assume that for every 2-vertex v in G the number of vertices at distance one or two from v is at least 5. So the neighbors of every 2-vertex are two nonadjacent 3-vertices or one neighbor of the 2-vertex is a 3-vertex and the other neighbor of it is a 2-vertex which is adjacent to another 3-vertex. We contract each edge incident with exactly one 2-vertex. Then there is no 2-vertex in the resulting graph H . After this operation, H still is an outerplanar graph but there is no 2-vertex in H now. Therefore we get a contradiction to Lemma 1.1.

Lemma 1.3 Let G be a 2-connected outerplanar graph with $\Delta(G) = 3$. Then $\chi(G^2) \leq \Delta(G) + 2$.

Proof We use induction method on the order of G . First, we consider the graph with order 4 and maximum degree 3. The lemma holds trivially. Now we assume that the lemma holds for all outerplanar graphs H with $|V(H)| < |V(G)|$. By Lemma 1.2, we find that there is a 2-vertex, say v , in G such that the number of vertices at distance one or two from v is at most 4. Let u and w be two neighbors of v . Let $H = G/uv$. H is still a 2-connected simple outerplanar graph with $\Delta(H) = 3$, and $\chi(H^2) \leq \Delta(H) + 2$. Since the number of vertices at distance one or two from v in G is at most 4 we can assign a suitable color for v in G^2 .

Lemma 1.4 Let G be a 2-connected outerplanar graph with $\Delta(G)=4$. Then there must be a 2-vertex v in G such that the number of vertices at distance one or two from v is at most 5.

Proof By contradiction. Assume that for every 2-vertex v in G the number of vertices at distance one or two from v is at least 6. Thus, all possible 2-vertices are included in the following five cases:

Case 1 There are two adjacent 2-vertices such that both the other neighbors of them are 4-vertices (see Fig. 1 (a));

Case 2 There is a 2-vertex such that the neighbors of it are a 3-vertex and a 4-vertex which are not adjacent (see Fig. 1 (b));

Case 3 There is a 2-vertex such that both the neighbors of it are nonadjacent 4-vertices (see Fig. 1 (c));

Case 4 There is a 2-vertex such that both the neighbors of it are adjacent 4-vertices (see Fig. 1 (d));

Case 5 There is a 2-vertex such that both the neighbors of it are nonadjacent 3-vertices (see Fig. 1 (e)).

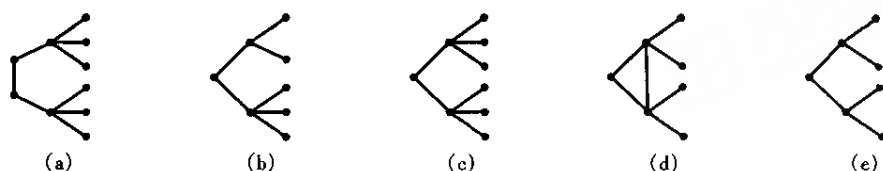


Fig.1 Five cases in Lemma 1.4

Next we construct a new graph H to get a contradiction. In Case 1, we delete the two adjacent 2-vertices and add a new edge between the two 4-vertices. In Case 2, 3, 5, we delete the 2-vertex and add a new edge to the two neighbors of the 2-vertex. In Case 4, we delete the 2-vertex. Obviously, H is still an outerplanar graph, but there is no 2-vertex in H now. This contraction completes the proof.

Lemma 1.5 Let G be a 2-connected outerplanar graph with $\Delta(G)=4$. Then $\chi(G^2) \leq \Delta(G)+2$.

Proof By Lemma 1.4, we can prove this lemma similarly to those of Lemma 1.3.

Lemma 1.6 (Weifan Wang[7]) If G is a 2-connected outerplanar graph with maximum degree $\Delta(G) \geq 5$, then at least one of the following conclusions is true:

(1) There exist two 2-vertices u and v such that $uv \in E(G)$;

(2) There exist three vertices u, v and w such that u is a 2-vertex, v is a 3-vertex and w is a ≥ 3 -vertex, and uvw is a 3-face;

(3) There exist three vertices u, v and w such that u is a 2-vertex, v and w are 4-vertices, and uvw is a 3-face;

(4) There exists a 4-vertex u , and $N_G(u) = \{u_1, u_2, v_1, v_2\}$ such that u_1 and u_2 are 2-vertices, v_1 and v_2 are ≥ 5 -vertices, and $u_1 v_1, u_2 v_2, v_1 v_2 \in E(G)$.

2 Main Results

Lemma 2.1 Let G be a 2-connected outerplanar graph with $\Delta(G) \geq 5$. Then $\chi(G^2) \leq \Delta(G)+2$.

Proof We will use induction method on the order of G . First, we concern the graph G with maximum degree Δ and with as less vertices as possible. Obviously, G is a fan with order $\Delta+1$. Thus the lemma holds trivially. From now on we assume that G is an outerplanar graph such that for all 2-connected outerplanar graphs H with $|V(H)| < |V(G)|$ the lemma is true. Next we prove this lemma also holds for G by considering the following four cases by Lemma 1.6:

Case 1 There exist three vertices u, v and w such that u is a 2-vertex, v is a 3-vertex and w is a ≥ 3 -vertex, and uvw is a 3-face. Let $H = G - u$. Since H is still a 2-connected outerplanar graph, there is a $(\Delta+2)$ -coloring c of H^2 . Because the number of vertices at distance one or two from u in G in total is at most $\Delta+1$. Then we can extend the coloring of H^2 to G^2 .

Case 2 There exists a 4-vertex u , and $N_G(u) = \{u_1, u_2, v_1, v_2\}$ such that u_1 and u_2 are 2-vertices, v_1 and v_2 are ≥ 5 -vertices, and $u_1 v_1, u_2 v_2, v_1 v_2 \in E(G)$.

v_2 are ≥ 5 -vertices, and $u_1 v_1, u_2 v_2, v_1 v_2 \in E(G)$. Let $H = G - u_2$. Then H is a 2-connected outerplanar graph, and $\Delta(H) = \Delta(G)$. By induction hypothesis, $\chi(H^2) \leq \Delta + 2$. Since the vertices at distance one or two from u_2 in G is at most $\Delta + 1$, we can extend the coloring of H^2 to G^2 .

Case 3 There exist three vertices u, v and w such that u is a 2-vertex, v and w are 4-vertices, and uvw is a 3-face. Let $H = G - u$. Then H is a 2-connected outerplanar graph with $\Delta(H) = \Delta(G)$. By induction hypothesis, $\chi(H^2) \leq \Delta + 2$. Since the vertices at distance one or two from u is 6 in G , we can extend the coloring of H^2 to G^2 .

Case 4 There exist two 2-vertices u and v such that $uv \in E(G)$. Let $x \neq v$ be the other neighbor of u and $y \neq u$ be the other neighbor of v . Without loss of generality we may assume that $xy \in E(G)$, otherwise it is contradicted to Lemma 1.1. Let $H = G - u - xv$. By induction hypothesis, there is a $(\Delta + 2)$ -coloring of H^2 . Since the total number of vertices at distance one or two from u is $\Delta + 1$ at most. We can extend the coloring of H^2 to a $(\Delta + 2)$ -coloring of G^2 . Finished the proof.

Lemma 2.2 Let G be a 2-connected outerplanar graph with $\Delta(G)$, $G \neq C_5$. Then $\chi(G^2) \leq \Delta(G) + 2$.

Proof It is easy to get the result from Lemma 1.3, 1.5, 2.1 and Theorem 1.1.

Theorem 2.1 Let G be an outerplanar graph with $\Delta(G)$, $G \neq C_5$. Then $\chi(G^2) \leq \Delta(G) + 2$.

Proof By Lemma 2.2 the Theorem holds when G is 2-connected. Without loss of generality, we may assume that there is just one cut vertex v whose degree is $\Delta(G)$ in G . Let G_1 and G_2 be the two blocks of G . Assume that $d_{G_1}(v) = k$, then $d_{G_2}(v) = \Delta(G) - k$. Let $N_{G_1}(v) = \{u \mid uv \in E(G_1)\}$ and $N_{G_2}(v) = \{u \mid uv \in E(G_2)\}$. By Lemma 2.2, both G_1^2 and G_2^2 are $(\Delta(G) + 2)$ -colorable. Let c_1 be a $(\Delta(G) + 2)$ -coloring of G_1^2 . Now we can give a $(\Delta(G) + 2)$ -coloring c_2 of G_2^2 such that $c_2(y) \neq c_1(x)$, $\forall y \in N_{G_2}(v)$, $x \in N_{G_1}(v)$ and $c_2(v) = c_1(v)$. Let $c(x) = c_1(x) \forall x \in V(G_1)$ and $c(x) = c_2(x) \forall x \in V(G_2)$. Obviously, we can verify that c is a $(\Delta(G) + 2)$ -coloring of G^2 . Thus, $\chi(G^2) \leq \Delta(G) + 2$.

Acknowledgements We would like to thank Dr. Xu Baogang deeply for his significantly useful suggestions. In addition, we would like to thank the anonymous referee for his patience, helpful criticism, too.

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外部平面图的平方图的染色

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[摘要] 图 G 的平方图, 记作 G^2 , 是一个以原图的顶点集为顶点集, 若原图中两点的距离不大于 2 则连以边所成的图. 本文确定了圈的平方图的色数. 对于外部平面图, 得到以下结论: 设 G 是一个最大度为 $\Delta(G)$ 的简单连通外部平面图, $G \neq C_5$. 则 $\chi(G^2) \leq \Delta(G) + 2$.

[关键词] 平方图, 色数, 外部平面图

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