Existence and Structure of Entire Explosive Positive Radial Solutions for a Class of Quasilinear Elliptic Systems

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Abstract: We study the existence and structure of entire explosive positive radial solutions for quasilinear elliptic systems $\operatorname{div}(|\nabla u|^{m-2}\nabla u) = p(|x|)f(v)$, $\operatorname{div}(|\nabla v|^{n-2}\nabla v) = q(|x|)g(u)$ on $\mathbf{R}^N$, where f and g are positive and non-decreasing functions on $(0,\infty)$. The main results of the present paper are new and extend the previously known results.

Key words: entire, explosive, positive radial solution, quasilinear elliptic system

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一类拟线性椭圆型方程组爆破整体正对称解的存在性和解的结构

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[摘要] 研究了拟线性椭圆型方程组 $\operatorname{div}(|\nabla u|^{m-2}\nabla u) = p(|x|)f(v)$, $\operatorname{div}(|\nabla v|^{n-2}\nabla v) = q(|x|)g(u)$ 在 $\mathbf{R}^N$ 上爆破整体正对称解的存在性和解集的性质,其中 f 和 g 在 $(0,\infty)$ 上是正的递增函数. 本文结果是新的且推广了所知结果.

[关键词] 整体，爆破,正对称解，拟线性椭圆型方程组

0 Introduction

Existence and non – existence of solutions of the quasilinear elliptic system

$$\begin{cases} \operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u,v) = 0, \ x \in \mathbf{R}^N \\ \operatorname{div}(|\nabla v|^{n-2}\nabla v) + g(u,v) = 0, \ x \in \mathbf{R}^N \end{cases} \tag{1}$$

has received much attention recently. See, for example, [2,4,5,7,10,11]. Problem (1) arises in the theory of quasi-regular and quasi-conformal mappings as well as in the study of non-Newtonian fluids. In the latter case, the pair (m,n) is a characteristic of the medium. Media with $(m,n) > (2,2)$ are called dilatant fluids and those with $(m,n) < (2,2)$ are called pseudoplastics. If $(m,n) = (2,2)$, they are Newtonian fluids.

When $m = n = 2$, system (1) becomes

$$\begin{cases} \triangle u + f(u,v) = 0, \ x \in \mathbf{R}^N \\ \triangle v + g(u,v) = 0, \ x \in \mathbf{R}^N \end{cases}$$

for which the existence and the non-existence of positive solutions and explosive positive solution has been inves-

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tigated extensively. We list here, for example, [1,3,6,8,9] and refer to the references therein.

When $m=n=2$, $f=-p(|x|)v^{\alpha}$, $g=-q(|x|)u^{\beta}$, system (1) becomes

$$\begin{cases}\triangle u=p(|x|)v^{\alpha},\ x\in\mathbf{R}^{N}\\ \triangle v=q(|x|)u^{\beta},\ x\in\mathbf{R}^{N}\end{cases}\tag{2}$$

for which existence results for entire explosive positive solutions can be found in a recent paper by Lair and Wood [6]. Lair and Wood established that all positive entire radial solutions of (2) are explosive provided that

$$\int_0^{\infty}tp(t)\,\mathrm{d}t=\infty,\ \int_0^{\infty}tq(t)\,\mathrm{d}t=\infty.$$

On the other hand, if

$$\int_0^{\infty}tp(t)\,\mathrm{d}t<\infty,\ \int_0^{\infty}tq(t)\,\mathrm{d}t<\infty$$

then all positive entire radial solutions of (2) are bounded.

F. Cirstea and V. D. Radulescu [1] and [11], extended the above results to a larger class of systems

$$\begin{cases}\triangle u=p(|x|)f(v),\ x\in\mathbf{R}^{N}\\ \triangle v=q(|x|)g(u),\ x\in\mathbf{R}^{N}\end{cases}$$

In this paper, we consider the following quasilinear elliptic system:

$$\begin{cases}\operatorname{div}(|\nabla u|^{m-2}\nabla u)=p(|x|)f(v),\ x\in\mathbf{R}^{N}\\ \operatorname{div}(|\nabla v|^{n-2}\nabla v)=q(|x|)g(u),\ x\in\mathbf{R}^{N},\end{cases}\tag{3}$$

where $N\geqslant 3$, $m>1$, $n>1$ and $p,q\in C(\mathbf{R}^{N})$ are positive functions, and satisfy the decay conditions

$$\int_0^{\infty}\left(t^{1-N}\int_0^{t}s^{N-1}p(s)\,\mathrm{d}s\right)^{1/(m-1)}\mathrm{d}t<\infty,\ \int_0^{\infty}\left(t^{1-N}\int_0^{t}s^{N-1}q(s)\,\mathrm{d}s\right)^{1/(n-1)}\mathrm{d}t<\infty.\tag{4}$$

We also assume that f, $g\in C[0,\infty)$ are positive, non-decreasing on $(0,\infty)$, and f and g also satisfy

$$\int_1^{\infty}\left[\int_0^{s}f(t)\,\mathrm{d}t\right]^{-(1/m)}\mathrm{d}s<\infty\ \text{and}\ \int_1^{\infty}\left[\int_0^{s}g(t)\,\mathrm{d}t\right]^{-(1/n)}\mathrm{d}s<\infty\tag{5}$$

In [10], we study the existence of entire explosive positive solutions of systems (3). In this paper, we obtain more results under new conditions. So the following results obtained complement corresponding results in [10] and extend the results in [1,6,11]. Using an argument inspired by Lair and Wood [6] and F. Cirstea and V. Radulescu [1], we obtain the following main results.

We use the notation $\mathbf{R}^{+}=[0,+\infty)$, and define the

$\mathscr{G}=\{(a,b)\in\mathbf{R}^{+}\times\mathbf{R}^{+}\mid u(0)=a,\ v(0)=b$, and (u,v) is an entire radial solution of $(3)\}$.

Theorem 1 Let $f,g\in C^{1}[0,\infty)$ satisfy (5) and $f(s)\leqslant g(s)$ for $s>0$, and the following condition

(H) for arbitrary nonnegative numbers c, d and $\lambda\in(0,1)$, the functions f and g satisfy

$$f[\lambda c+(1-\lambda)d]\leqslant\lambda f(c)+(1-\lambda)f(d),\ g[\lambda c+(1-\lambda)d]\leqslant\lambda g(c)+(1-\lambda)g(d).$$

Assume (4) holds, $\eta(|x|)=\min\{p(|x|),q(|x|)\}\geqslant C>0$. Then set $\mathscr{G}\neq\varnothing$ and is a closed bounded convex subset of $\mathbf{R}^{+}\times\mathbf{R}^{+}$. Furthermore, the set G satisfies

$$T\subset G\subset R,\tag{6}$$

where the triangle T and the rectangle R are given by $T=\{(u,v)\in\mathbf{R}^{+}\times\mathbf{R}^{+}:\frac{u}{A}+\frac{v}{B}\leqslant 1\}$, $R=[0,A]\times[0,B]$, in which $A=\sup\{a\in\mathbf{R}^{+}\mid(a,0)\in G\}$ and $B=\sup\{b\in\mathbf{R}^{+}\mid(0,b)\in G\}$.

Theorem 2 Let $f,g\in C^{1}[0,\infty)$ satisfy (5) and $f(s)\leqslant g(s)$ for $s>0$. Assume (4) holds, $\eta(|x|)\geqslant C>0$ and $v=\max\{n(0),n(0)\}>0$. Let $E(\mathscr{G})$ be the closure of the set $\{(a,b)\in\partial\mathscr{G}\mid a>0,\ b>0\}$. Then any entire positive radial solution (u,v) of (3) with central value $(u(0),v(0))\in E(\mathscr{G})$ is explosive.

Remark 1 If $N\geqslant 3$, $m,n<N$, then condition (4) is replaced by

$$0<\int_1^{\infty}r^{\frac{1}{m-1}}p(r)^{\frac{1}{m-1}}\mathrm{d}r<\infty,\ 0<\int_1^{\infty}r^{\frac{1}{n-1}}q(r)^{\frac{1}{n-1}}\mathrm{d}r<\infty\ \text{if}\ 1<m,n\leqslant 2,\tag{A}$$

$$0<\int_1^{\infty}r^{\frac{(m-2)N+1}{m-1}}p(r)\,\mathrm{d}r<\infty,\ 0<\int_1^{\infty}r^{\frac{(n-2)N+1}{n-1}}q(r)\,\mathrm{d}r<\infty\ \text{if}\ m,n\geqslant 2.\tag{B}$$

Let

$$J(r) = \int_0^r \left(t^{1-N}\int_0^t s^{N-1}p(s)\,ds\right)^{\frac{1}{m-1}}dt$$

If fact, if $1 < m \leqslant 2$, by estimating above the integral

$$J(r) \leqslant C_1 + \int_1^r t^{\frac{1-N}{m-1}}\left[\int_0^t s^{N-1}p(s)\,ds\right]^{1/(m-1)}dt.$$

Using the assumption $N \geqslant 3$ in the computation of the first integral above and Jensen's inequality to estimate the last one,

$$J(r) \leqslant C_2 + C_3\int_1^r t^{\frac{3-N-m}{m-1}}\int_1^t s^{\frac{N-1}{m-1}}p(s)^{\frac{1}{m-1}}ds\,dt.$$

Computing the above integral, we obtain

$$J(r) \leqslant C_2 + C_4\int_1^r t^{\frac{1}{m-1}}p(t)^{\frac{1}{m-1}}dt.$$

Applying (A) in the integral above we infer that $H_\infty = \lim_{r\to\infty} J(r) < \infty$. On the other hand, if $m \geqslant 2$, set

$$H(t) = \int_0^t s^{N-1}p(s)\,ds$$

and note that either, $H(t) \leqslant 1$ for $t > 0$ or $H(t_0) = 1$ for some $t_0 > 0$. In the first case, $H^{\frac{1}{m-1}} \leqslant 1$, and hence,

$$J(r) = \int_0^r t^{\frac{1-N}{m-1}}H(t)^{\frac{1}{m-1}}dt \leqslant C_5 + \int_1^r t^{\frac{1-N}{m-1}}dt$$

so that $J(r)$ has a finite limit because $m < N$. In the second case, $H(s)^{\frac{1}{m-1}} \leqslant H(s)$ for $s \geqslant s_0$ and hence,

$$J(r) \leqslant C_6 + \int_1^r t^{\frac{1-N}{m-1}}\int_0^s s^{N-1}p(s)\,ds\,dt.$$

Estimating and integrating by parts, we obtain

$$\begin{aligned} J(r) &\leqslant C_6 + C_7\int_1^r t^{\frac{1-N}{m-1}}dt + \frac{m-1}{N-m}\left[\int_1^r t^{\frac{(m-2)N+1}{m-1}}p(t)\,dt - r^{\frac{m-N}{m-1}}\int_0^t t^{N-1}p(t)\,dt\right] \\ &\leqslant C_8 + C_9\int_1^r t^{\frac{(m-2)N+1}{m-1}}p(t)\,dt. \end{aligned}$$

By (B), $H_\infty = \lim_{r\to\infty} J(r) < \infty$. Other second Eq. of condition (4) similarly.

Remark 2 If condition $\eta(|x|) = \min\{p(|x|), q(|x|)\} \geqslant C > 0$ is replaced by $\eta(x)$ is non-negative on $\Omega \subseteq \mathbf{R}^N$ and satisfies the following: if $x_0 \in \Omega$ and $\eta(x_0) = 0$, then there exists a domain Ω_0 such that $x_0 \in \Omega_0 \subset \Omega$ and $\eta(x) > 0$ for all $x \in \partial\Omega_0$. then the conclusions of Theorems 1 ~ 2 still hold.

1 Preliminary Results

In this section we consider some preliminary results for quasilinear elliptic equation

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) = p(x)f(u),\ x \in \mathbf{R}^N (N \geqslant 2), \tag{7}$$

where $m > 1$, $\nabla u = (\nabla_1 u, \cdots, \nabla_N u)$, $p(x): \mathbf{R}^N \to (0, \infty)$ and $f: (0, \infty) \to (0, \infty)$ are continuous functions. A positive entire solution of the equation (7) is defined to be a positive function $u \in C^1(\mathbf{R}^N)$ satisfying (7) at every point of $\mathbf{R}^N$.

From reference [10], we give the following lemma

Lemma 1 (Weak Comparison Principle). Let Ω be a bounded domain in $\mathbf{R}^N (N \geqslant 2)$ with smooth boundary $\partial\Omega$ and $\theta: (0, \infty) \to (0, \infty)$ is continuous and nondecreasing. Let $u_1, u_2 \in W^{1,m}(\Omega)$ satisfy

$$\int_\Omega |\nabla u_1|^{m-2}\nabla u_1 \nabla\psi\,dx + \int_\Omega \theta(u_1)\psi\,dx \leqslant \int_\Omega |\nabla u_2|^{m-2}\nabla u_2\nabla\psi\,dx + \int_\Omega \theta(u_2)\psi\,dx$$

for all non-negative $\psi \in W_0^{1,m}(\Omega)$. Then the inequality

$$u_1 \leqslant u_2 \text{ on } \partial\Omega$$

implies that

$$u_1 \leqslant u_2 \text{ in } \Omega.$$

Lemma 2 If $f(u)$ satisfies (5) and $p(x) \geqslant C > 0$, then in any bounded domain D there exists a solution of (7) which becomes infinite on S.

Lemma 3 Suppose that f satisfies (5), $f \in C^1[0,\infty)$, $f(0)=0$ and $p(|x|) \geqslant C > 0$ for $x \in \mathbf{R}^N$ and the following:

$$\int_1^\infty r^{\frac{1}{m-1}} p(r)^{\frac{1}{m-1}} \mathrm{d}r < \infty \text{ if } 1 < m,n \leqslant 2$$

$$\int_1^\infty r^{\frac{(m-2)N+1}{m-1}} p(r) \mathrm{d}r < \infty \text{ if } m,n \geqslant 2$$

Then Eq.

$$\operatorname{div}(|\nabla u|^{m-2} \nabla u) = p(|x|) f(u)$$

has an entire explosive positive radial solution.

Lemma 4 The problem

$$\operatorname{div}(|\nabla l|^{m-2} \nabla l) = (p(|x|) + q(|x|))(f(l) + g(l)) \tag{8}$$

and

$$\operatorname{div}(|\nabla h|^{n-2} \nabla h) = (p(|x|) + q(|x|))(f(h) + g(h)) \tag{9}$$

has an entire explosive positive radial solution provided that functions $\eta(|x|) \geqslant C > 0$ satisfy (4) and f,g satisfy (5) and $f(s) \leqslant g(s)$.

Proof From lemma 2, for each natural number k, let v_k be a positive solution of the boundary-value problem

$$\operatorname{div}(|\nabla v_k|^{m-2} \nabla v_k) = (p(|x|) + q(|x|))(f(v_k) + g(v_k)), \quad |x| < k,$$
$$v_k \to \infty, \quad |x| \to k.$$

Again, from Lemma 1, we can show that

$$v_1 \geqslant v_2 \geqslant \cdots \geqslant v_k \geqslant v_{k+1} \geqslant \cdots > 0$$

in $\mathbf{R}^N$. To complete the proof, it is sufficient to show that there exists a function $w \in C(\mathbf{R}^N)$ such that $w \to \infty$ as $|x| \to \infty$ and $v_k \geqslant w$ in $\mathbf{R}^N$ for all k. To do this, we note first that condition $f(s) \leqslant g(s)$, and we consider the equation

$$\operatorname{div}(|\nabla u|^{m-2} \nabla u) = [p(|x|) + q(|x|)] f(u). \tag{10}$$

By Lemma 3, Eq (10) has a positive solution u on $\mathbf{R}^N$ such that $u(x) \to \infty$ as $|x| \to \infty$. We claim that $w = u - 1$ is a desired lower boundary for v_k. Indeed, since

$$\operatorname{div}(|\nabla(v_k+1)|^{m-2} \nabla(v_k+1)) = \operatorname{div}(|\nabla v_k|^{m-2} \nabla v_k) = (p+q)(f(v_k) + g(v_k))$$
$$\leqslant (p+q)(f(v_k+1) + g(v_k+1)) \leqslant (p+q) f(v_k+1) \text{ for } |x| > k,$$

and clearly $v_k + 1 > u$ as $|x| \to k$, Lemma 1 implies that $v_k + 1 \geqslant u$ for $|x| \leqslant k$. Hence $v = \lim_{k\to\infty} v_k \geqslant u - 1$ on $\mathbf{R}^N$. Again, by the standard regularity argument for elliptic problems, it is a straight forward argument to prove that v is the desired solution of (8). By a similar argument, we can show that (9) has an entire explosive positive radial solution.

Lemma 5 Suppose g_R, h_R are positive radial solutions of the problem

$$\operatorname{div}(|\nabla g_R|^{m-2} \nabla g_R) = p(r) f(g_R) + q(r) g(g_R) \quad 0 \leqslant r < R$$
$$g_R(r) \to \infty \quad r \to R^-$$

and

$$\operatorname{div}(|\nabla h_R|^{n-2} \nabla h_R) = p(r) f(h_R) + q(r) g(h_R) \quad 0 \leqslant r < R$$
$$h_R(r) \to \infty \quad r \to R^-$$

where p and q are non-negative $C([0,\infty))$ functions. Then $\lim_{R\to 0^+} g_R(0) = \infty$, and $\lim_{R\to 0^+} h_R(0) = \infty$.

Proof Since $g'_R(r) \geqslant 0$ and p,q are bounded on $[0,1]$, we get

$$(g'_R(r))^{m-1} = r^{1-N} \int_0^r s^{N-1} [p(s) f(g_R)(s) + q(s) g(g_R)(s)] \mathrm{d}s$$

$$\leqslant \int_0^r [p(s)f(g_R)(s) + q(s)g(g_R)(s)]\,ds \leqslant af(g_R)(r) + bg(g_R)(r),$$

then

$$g'_R(r) \leqslant (af(g_R)(r) + bg(g_R)(r))^{1/(m-1)} \leqslant$$
$$\begin{cases} a^{1/(m-1)}f(g_R)^{1/(m-1)}(r) + b^{1/(m-1)}g(g_R)^{1/(m-1)}(r) \text{ for } m \geqslant 2 \\ 2^{(2-m)/(m-1)}(a^{1/(m-1)}f(g_R)^{1/(m-1)}(r) + b^{1/(m-1)}g(g_R)^{1/(m-1)}(r)) \text{ for } 1 < m < 2, \end{cases}$$

where

$$a = \int_0^1 p(s)\,ds,\ b = \int_0^1 q(s)\,ds.$$

Thus we have

$$\begin{cases} -\dfrac{d}{dr}\displaystyle\int_{g_R(r)}^{\infty} \frac{ds}{a^{1/(m-1)}f^{1/(m-1)}(s) + b^{1/(m-1)}g^{1/(m-1)}(s)} \leqslant 1 \text{ for } m \geqslant 2 \\ \dfrac{d}{dr}\displaystyle\int_{g_R(r)}^{\infty} \frac{ds}{2^{(2-m)/(m-1)}(a^{1/(m-1)}f^{1/(m-1)}(s) + b^{1/(m-1)}g^{1/(m-1)}(s))} \leqslant 1 \text{ for } 1 < m < 2, \end{cases}$$

Now integrating from 0 to R, and recalling that $g_R(r) \to \infty$ as $r \to R^-$ we get

$$\begin{cases} \displaystyle\int_{g_R(0)}^{\infty} \frac{ds}{a^{1/(m-1)}f^{1/(m-1)}(s) + b^{1/(m-1)}g^{1/(m-1)}(s)} \leqslant R \text{ for } m \geqslant 2 \\ \displaystyle\int_{g_R(0)}^{\infty} \frac{ds}{2^{(2-m)/(m-1)}(a^{1/(m-1)}f^{1/(m-1)}(s) + b^{1/(m-1)}g^{1/(m-1)}(s))} \leqslant R \text{ for } 1 < p < 2. \end{cases}$$

Letting $R \to 0^+$ yields

$$\lim_{R \to 0^+} \int_{g_R(0)}^{\infty} \frac{ds}{a^{1/(m-1)}f^{1/(m-1)}(s) + b^{1/(m-1)}g^{1/(m-1)}(s)} = 0.$$

Hence we have $g_R(0) \to \infty$ as $R \to 0^+$. By a similar argument, we can show that $h_R(0) \to \infty$ as $R \to 0^+$.

By similar argument with Lemma 6 of [6], it is easy to prove the following lemma

Lemma 6 Let l, h be any entire explosive positive radial solution of (8) ~ (9) given in Lemma 6 and define the sequences $\{u_k\}$ and $\{v_k\}$ by

$$u_k(r) = a + \int_0^r (t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\,ds)^{1/(m-1)}\,dt,\ r \geqslant 0,$$

$$v_k(r) = b + \int_0^r (t^{1-N}\int_0^t s^{N-1}q(s)f(u_{k-1}(s))\,ds)^{1/(n-1)}\,dt,\ r \geqslant 0,$$

where $u_0 = a$, $0 \leqslant a \leqslant \min\{l(0), h(0)\}$ and $v_0(r) = b$, $0 \leqslant b \leqslant \min\{l(0), h(0)\}$. Then

(a) $u_k(r) \leqslant u_{k+1}(r)$ and $v_k(r) \leqslant v_{k+1}(r)$, $r \in \mathbf{R}^+$, $k \geqslant 1$, and

(b) $u_k(r) \leqslant l(r)$ and $v_k(r) \leqslant h(r)$, $r \in \mathbf{R}^+$, $k \geqslant 1$.

Thus $\{u_k\}$ and $\{v_k\}$ converge and the limit functions are entire positive radial solutions of system (3).

2 Proof of Main Theorems

Proof of Theorem 1 Since the radial solutions of (3) are solutions of the ordinary differential equations system

$$(r^{N-1}|u'|^{m-2}u')' = r^{N-1}p(r)g(v(r)),\ (r^{N-1}|v'|^{n-2}v')' = r^{N-1}q(r)f(u(r)) \text{ for } r > 0$$

it follows that the radial solutions of (3) with $u(0) = a > 0$, $v(0) = b > 0$ satisfy:

$$u(r) = a + \int_0^r (t^{1-N}\int_0^t s^{N-1}p(s)g(v(s))\,ds)^{1/(m-1)}\,dt,\ r \geqslant 0, \tag{11}$$

$$v(r) = b + \int_0^r (t^{1-N}\int_0^t s^{N-1}q(s)f(u(s))\,ds)^{1/(n-1)}\,dt,\ r \geqslant 0. \tag{12}$$

From Lemma 6, it is clear that $[0, g(0)] \times [0, h(0)] \subset \mathscr{G}$ so that $\mathscr{G}$ is non-empty. We shall show that $\mathscr{G}$ is a bounded, closed set.

As a preliminary, note that if $(a, b) \in \mathscr{G}$ then any pair (a_0, b_0) for which $0 \leqslant a_0 \leqslant a$ and $0 \leqslant b_0 \leqslant b$ must be in $\mathscr{G}$ since the process used in Lemma 7 can be repeated with

$$u_k(r) = a_0 + \int_0^r \left(t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\mathrm{d}s\right)^{1/(m-1)}\mathrm{d}t,$$

$$v_k(r) = b_0 + \int_0^r \left(t^{1-N}\int_0^t s^{N-1}q(s)f(u_{k-1}(s))\mathrm{d}s\right)^{1/(n-1)}\mathrm{d}t,$$

and $v_0 = b$, $u_0 = a$. Then, as in Lemma 6, the sequences $\{u_k\}$ and $\{v_k\}$ are monotonically increasing. Then, letting (U,V) be the solution of (11) and (12) with central values (a,b), we can easily prove, since $b_0 \leqslant b$, that $v_0 \leqslant V$. Thus, $u_1 \leqslant U$ (since, also, $a_0 \leqslant a$), and consequently $v_1 \leqslant V$, and so on. Hence we get $u_k \leqslant U$ and $v_k \leqslant V$, and therefore $u \leqslant U$ and $v \leqslant V$ where $(u,v) = \lim\limits_{k\to\infty}(u_k, v_k)$ is a solution of (3) (with central values (a_0, b_0)).

Lemma 2 ensures the existence of a positive explosive solution h_1, h_2 of the problem

$$\mathrm{div}(|\nabla h_1|^{m-2}\nabla h_1) = \eta(|x|)g(h_1) \text{ in } B(0,R)$$
$$h_1 \to \infty \quad |x| \to R$$
$$\mathrm{div}(|\nabla h_2|^{n-2}\nabla h_2) = \eta(|x|)g(h_2) \text{ in } B(0,R)$$
$$h_2 \to \infty \quad |x| \to R.$$

To prove that $\mathscr{G}$ is bounded, assume that it is not. Then, there exists $(a,b) \in \mathscr{G}$ such that $a + b > \max\{2\delta, h_1(0) + h_2(0)\}$. Let (u,v) be the entire radial solution of (3) such that $(u(0), v(0)) = (a,b)$. Since $u(x) + v(x) \geqslant a + b > 2\delta$ for all $x \in \mathbf{R}^N$, by $f(s) \leqslant g(s)$ we get:

$$\mathrm{div}(|\nabla u|^{m-2}\nabla u) = p(|x|)g(v) \geqslant \eta(|x|)g(v),\ \mathrm{div}(|\nabla v|^{n-2}\nabla v) = q(|x|)f(u) \geqslant \eta(|x|)g(u).$$

On the other hand, $h_1(x) \to \infty$, $h_2(x) \to \infty$ as $|x| \to R$. Thus, Lemma 1 we conclude that $u + v \leqslant h_1 + h_2$ in $B(0,R)$. But this is impossible since $u(0) + v(0) = a + b > h_1(0) + h_2(0)$.

To prove that $\mathscr{G}$ is closed, we let $(a_0, b_0) \in \partial\mathscr{G}$ and show that $(a_0, b_0) \in \mathscr{G}$. Let (u,v) be the solution of (11) and (12) which corresponds to $a = a_0$ and $b = b_0$. Without loss of generality, we may assume that $\max\{a_0, b_0\} > C = l(0)$ where the function l is given in Lemma 7. If $\max\{a_0, b_0\} = a_0$, then $C \leqslant a_0 - 1/k$ for large k so that $u_k(r) \geqslant C$ for all $r \geqslant 0$ and for all k sufficiently large where

$$u_k = a_0 - \frac{1}{k} + \int_0^r \left(t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\mathrm{d}s\right)^{1/(m-1)}\mathrm{d}t,$$

$$v_k = b_0 + \int_0^r \left(t^{1-N}\int_0^t s^{N-1}q(s)f(u_{k-1}(s)\mathrm{d}s\right)^{1/(n-1)}\mathrm{d}t.$$

From (11), we have

$$\mathrm{div}(|\nabla u_k|^{m-2}\nabla u_k) \geqslant \eta(r)g(v_k),\ \mathrm{div}(|\nabla v_k|^{n-2}\nabla v_k) \geqslant \eta(r)g(u_k).$$

Let $h_1(r)$, $h_2(r)$ are positive solution of

$$\mathrm{div}(|\nabla h_1|^{m-2}\nabla h_1) = \eta(r)g(h_1),\ 0 \leqslant r < R_0,$$
$$h_1(r) \to \infty,\ r \to R_0^-,$$

and

$$\mathrm{div}(|\nabla h_2|^{n-2}\nabla h_2) = \eta(r)g(h_2),\ 0 \leqslant r < R_0,$$
$$h_2(r) \to \infty,\ r \to R_0^-,$$

where R_0 is an arbitrary positive real number. It is now easy to show by Lemma 1 that $u_k + v_k \leqslant h_1 + h_2$ in $[0, R_0]$. Hence $u + v = \lim\limits_{k\to\infty}(u_k + v_k) \leqslant h_1 + h_2$ on $[0, R_0]$. Since R_0 is arbitrary, the functions u, v exist on $\mathbf{R}^N$ and hence are entire so that $(a_0, b_0) \in \mathscr{G}$. On the other hand, if $\max\{a_0, b_0\} = b_0$, then $C \leqslant b_0 - 1/k$ for large k so that $v_k \geqslant C$ for all $r \geqslant 0$ and for all sufficiently large k. Then $u_k(r) \geqslant C^\alpha A(r)$ where $A(r) = \int_0^r \left(t^{1-N}\int_0^t s^{N-1}p(s)\mathrm{d}s\right)^{1/(m-1)}\mathrm{d}t$ and the proof continues as before with C replaced by $C^\alpha A(r)$.

To prove that $\mathscr{G}$ is convex, suppose $(a,b) \in G$ and $(\hat{a}, \hat{b}) \in G$. Let $\lambda \in (0,1)$, let (u,v) be the solution of (11) and (12), and let (U,V) be the solution of (11) and (12) when (a,b) is replace by $(\hat{a}, \hat{b})$. We need to prove that $\lambda(a,b) + (1-\lambda)(\hat{a}, \hat{b}) \in G$. To do this, we let $\{u_n\}$, $\{v_n\}$, $\{U_n\}$ and $\{V_n\}$ be the increasing sequences of functions, as developed in Lemma 4, such that u_n/u, v_n/v, U_n/U and V_n/V. Likewise, let $\{w_n\}$

and $\{z_n\}$ be the sequences developed again as in Lemma 4 corresponding to central values $\lambda a+(1-\lambda)\hat{a}$ and $\lambda b+(1-\lambda)\hat{b}$, respectively. We also let $z_0=\lambda b+(1-\lambda)\hat{b}$. We shall show that the increasing sequences $\{w_n\}$ and $\{z_n\}$ satisfy

$$w_n\leqslant\lambda u_n+(1-\lambda)U_n,\ z_n\leqslant\lambda v_n+(1-\lambda)V_n, \tag{13}$$

which, in turn, implies that $\{w_n\}$ and $\{z_n\}$ converge and hence their limits are entire, giving $\lambda(a,b)+(1-\lambda)\cdot(\hat{a},\hat{b})\in G$. Clearly $z_0\leqslant\lambda v_0+(1-\lambda)V_0$. We also have $f(\lambda v_0+(1-\lambda)V_0)\leqslant\lambda f(v_0)+(1-\lambda)f(V_0)$ and $g(\lambda v_0+(1-\lambda)V_0)\leqslant\lambda g(v_0)+(1-\lambda)g(V_0)$ by condition (H). Then

$$\begin{aligned}
w_1(r) &= \lambda a+(1-\lambda)\hat{a}+\int_0^r\left(t^{1-N}\int_0^t s^{N-1}p(s)f(z_0(s))\,ds\right)^{1/(m-1)}dt\\
&\leqslant \lambda a+(1-\lambda)\hat{a}+\int_0^r\left(t^{1-N}\int_0^t s^{N-1}p(s)f[\lambda v_0+(1-\lambda)V_0]\,ds\right)^{1/(m-1)}dt\\
&\leqslant \lambda a+(1-\lambda)\hat{a}+\int_0^r\left(t^{1-N}\int_0^t s^{N-1}p(s)[f(\lambda v_0)+(1-\lambda)f(V_0)]\,ds\right)^{1/(m-1)}dt\\
&\leqslant \lambda a+(1-\lambda)\hat{a}+\int_0^r\int_0^t\left(\frac{s}{t}\right)^{(N-1)/(m-1)}p^{1/(m-1)}(s)[f^{1/(m-1)}(\lambda v_0)+(1-\lambda)f^{1/(m-1)}(V_0)]\,ds\,dt\\
&= \lambda u_1+(1-\lambda)U_1.
\end{aligned}$$

Using this result, we can prove similarly that $z_1\leqslant\lambda v_1+(1-\lambda)V_1$ which, in ture, can be used to get $w_2\leqslant\lambda u_2+(1-\lambda)U_2$. Continuing this process will produce (13).

To prove (6), it is clear that since $(A,0)$ and $(0,B)$ are in G and G is convex, the line $\frac{x}{A}+\frac{y}{B}=1$ is in G. And, as noted earlier, if $(a,b)\in G$, then $(x_0,y_0)\in G$ whenever $0\leqslant x_0\leqslant a$ and $0\leqslant y_0\leqslant b$. Hence $T\subseteq G$. Similarly, $G\subseteq R$ for if $(a_0,b_0)\in G$, then $(a_0,0)\in G$ and $(0,b_0)\in G$. Thus $0\leqslant a_0\leqslant A$ and $0\leqslant b_0\leqslant B$ so that $(a_0,b_0)\in R$. This completes the proof.

Proof of Theorem 2 The proof is similar to the Theorem 2 of [1, 6, 10], so we omit the detail.

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