The Cauchy Problem and the Initial Boundary Value Problem for the Homogeneous GBBM Burgers Equations of Higher Order

Wei Suqing, Xu Guixiang

(The Graduate School of China Academy of Engineering Physics, 100088, Beijing, China)

Abstract: In this paper, we consider the Cauchy problem and the initial boundary value problem for the homogeneous GBBM Burgers equations of higher order. For any bounded or unbounded smooth domain, the existence and uniqueness of global strong solution are established by using Banach-fixed point theorem and a priori estimates for the Cauchy problem and IBV problem of homogeneous GBBM Burg-ers equations of higher order in $W^{\mu,p}(\Omega)$.

Key words: GBBM Burgers equations, pseudodifferential operator, Banach-fixed point theorem

CLC number O29, **Document code** A, **Article ID** 1001-4616(2005)01-0011-08

高阶齐次 GBBM Burgers 方程的 Cauchy 问题和初边值问题

魏素青, 徐桂香

(中国工程物理研究院,100088,北京)

[摘要] 本文中我们考虑了高阶齐次 GBBM Burgers 方程的 Cauchy 问题和初边值问题. 对任意有界或无界区域,使用 Banach 不动点定理和解的先验估计得到整体强解的存在与惟一性.

[关键词] GBBM Burgers 方程, 拟微分算子, Bannch 不动点, 定理

0 Introduction

In this paper, we discuss homogeneous generalized Benjamin-Bona-Mahony Burgers equation

$$Mu_t = \triangle u + (b, \nabla u) + u^r(a, \nabla u), \tag{1}$$

which satisfies the IBV conditions

$$\begin{cases} u(x, 0) = u_0(x), & x \in \Omega; \\ D^\alpha u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \ |\alpha| \leqslant [\mu] - 1 \end{cases} \tag{2}$$

or Cauchy condition

$$u(x, 0) = u_0(x), \quad x \in \mathbf{R}^n \tag{3}$$

Here Ω is a bounded or unbounded smooth domain in $\mathbf{R}^n$, $n \geqslant 1$, a, $b \in \mathbf{R}^n$, $r \in \boldsymbol{R}$, M is a pseudodifferential operator of order $\mu \geqslant 2$. $[\mu]$ denotes the integral part of μ. We assume that the operator M has the form

$$(\xi) = m(\xi)\hat{u}(\xi); \tag{4}$$

where the symbol m is a real even function with the following properties

$$C_1(1 + |\xi|)^\mu \leqslant |m(\xi)| \leqslant C_2(1 + |\xi|)^\mu, \quad (C_1, C_2 > 0). \tag{5}$$

Received date: 2004-09-02.
Foundation item: Supported by National Natural Science Foundation of China (104410021).
Biography: Wei Suqing, famale, born in 1977, master, Majored in Applied Mathematics. E-mail: weisuqing1207@sina.com

The classical BBM equation

$$u_t - u_{xxt} + \left(u + \frac{1}{2}u^2\right)_x = 0 \tag{6}$$

and the following GBBM equation in arbitrary dimonsions

$$u_t - \Delta u_t + \mathrm{div}\phi(u) = 0 \tag{7}$$

have been quite extensively studied in several papers [2—4,6,8,13—15,17]. Their main results can be stated as the following:

(1) For any bounded smooth domain Ω and $\max\left(1, \frac{n}{2}\right) \leqslant p < \infty$, there is a unique global strong solution for the problem (7),(2) or (7), (3) in $W^{2,p}(\Omega)$;

(2) For any unbounded smooth domain Ω or $\Omega = \mathbf{R}^n$ and $\max\left(1, \frac{n}{2}\right) \leqslant p < n$, there is a unique global strong solution for the problem (7), (2) or (7), (3) in $W^{2,p}(\Omega)$.

In [13] and [14] Miao Changxing have studied the inhomogeneous GBBM equation

$$u_t - \Delta u_t + \mathrm{div}\phi(u) = f(u), \tag{8}$$

and

$$u_t + (-1)^m \sum_{|\nu| \leqslant m} D^{2\nu} u_t - \sum_{|\alpha| \leqslant m, |\beta| \leqslant m} D^{\alpha}(a_{\alpha\beta}(x,t) D^{\beta}(u)) - \mathrm{div}\phi(u) = f(u), \tag{9}$$

and improved the known results, even the case of GBBM equations. This paper establishes the existence and uniqueness of global strong solution for the IBV problem and Cauchy problem of homogeneous GBBM Burgers equations in $W^{\mu,p}(\Omega)$ in the case $\max\left(1, \frac{n}{\mu}\right) \leqslant p < \infty$.

Notation In this paper Ω is a bounded or unbounded smooth domain in $\mathbf{R}^n$ or $\Omega = \mathbf{R}^n$, $L_p(\Omega)$ is Lebesgue space, $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ are Sobolev spaces. $\|\cdot\|_p$, $\|\cdot\|_{k,p}$ denote the $L_p(\Omega)$ norms and $W^{k,p}(\Omega)$ norms separately. Generic positive constants will be denoted by C; they do not depend on x and t, they may depend sometimes on u_0 and they may vary from line to line.

1 Main results

Let $\max\left(1, \frac{n}{\mu}\right) \leqslant p < \infty$, we define the pseudodifferential operator M with the domain $D(M) = W^{\mu,p}(\Omega) \cap W_0^{\frac{\mu}{2},p}(\Omega)$. By (4) and (5), we know M has an inverse operator and M^{-1} is a bounded linear operator from $L_p(\Omega)$ into $D(M)$. The $D(M)$ becomes a Banach space with respect to the graph norm $\|Mu\|_p$. It is well known that the graph norm is equivalent to $W^{\mu,p}(\Omega)$ norm. Hence we shall use $W_{\mu,p}(\Omega)$ norm on the Banach space $D(M)$ in this paper. By Sobolev embeding theorem we know $D(M)$　　$L_\infty(\Omega)$. Thus for any $u \in D(M)$, we have $\Delta u \in L_p(\Omega)$, $(b, \nabla u) \in L_p(\Omega)$, $u^r(a, \nabla u) \in L_p(\Omega)$. So it is easy to get the following lemma.

Lemma 1.1 Let $T > 0$, $\max\left(1, \frac{n}{\mu}\right) \leqslant p < \infty$, if $u \in C([0,T], D(M))$, then $M^{-1}\Delta u$, $M^{-1}(b, \nabla u)$, $M^{-1}u^r(a, \nabla u) \in C([0,T], D(M))$.

Definition 1.1 Let $T > 0$, $\max\left(1, \frac{n}{\mu}\right) \leqslant p < \infty$, $u_0 \in D(M)$, we call the function $u(x,t) \in C([0,T]; D(M))$ a strong solution of the problem (1), (2) or (1), (3) provided that $u(x,t)$ satisfies the following integral equation

$$u(t) = u_0(x) + \int_0^t M^{-1}\Delta u \mathrm{d}s + \int_0^t M^{-1}(b, \nabla u)\mathrm{d}s + \int_0^t M^{-1}u^r(a, \nabla u)\mathrm{d}s, \tag{10}$$

where

$$u(t) = u(x,t).$$

The main results of this paper are the following theorems.

Thereom 1.1 Assume Ω is bounded or unbounded smooth domain in $\mathbf{R}^n$, $\max(1,\frac{n}{\mu})\leqslant p<\infty$, and the conditions of Lemma 1.1 are satisfied, then for any $u_0\in D(M)$ there exists a time interval $[0,T]$ and a unique strong solution $u(x,t)$ of problem (1), (2) or (1), (3) on $[0,T]$.

Thereom 1.2 Assume that the conditions of Lemma 1.1 are satisfied. Let $q\in(1,\infty)$, we have

(1) If $u_0(x)\in D(M)\cap W^{\frac{\mu}{2},q}(\Omega)$, then there exists a time interval $[0,T]$ and a unique strong solution $u(t)$ of problem (1), (2) or (1), (3) belonging to the class $C^1([0,T];D(M)\cap W^{\frac{\mu}{2},q}(\Omega))$.

(2) If $u_0(x)\in D(M)\cap W^{\mu,q}(\Omega)$, then $u(t)\in C^1([0,T];D(M)\cap W^{\mu,q}(\Omega))$.

Thereom 1.3 Assumed that the conditions of Theorem 1.1 are satisfied, then for any $u_0\in D(M)\cap W^{\frac{\mu}{2},2}(\Omega)$, we have the following results

(1) When $n<\mu$, there is a unique global strong solution $u(t)$ for the problem (1), (2) or (1), (3) for all $p\geqslant 1$;

(2) When $n\geqslant\mu$, if

$$\begin{aligned} &0\leqslant r<\infty, && n=\mu;\\ &0\leqslant r\leqslant\frac{2(\mu-1)}{n-\mu}, && n>\mu; \end{aligned}\tag{11}$$

there is a unique global strong solution $u(t)$ for the problem (1), (2) or (1), (3) for all $\frac{n}{\mu}\leqslant p<\frac{n}{\mu-1}$;

(3) If $|\Omega|<\infty$, we have the same results as (2) for all $\frac{n}{\mu-1}\leqslant p<\infty$.

Thereom 1.4 Let $n\geqslant\mu$, Ω is an unbounded smooth domain or $\Omega=\mathbf{R}^n$. If

$$\begin{aligned} &\frac{2}{n}\leqslant r<\infty, && n=\mu,\\ &\frac{2}{n}\leqslant r\leqslant\frac{2}{n-\mu}, && n>\mu, \end{aligned}\tag{12}$$

then for any $u_0\in D(M)\cap W^{\frac{\mu}{2},2}(\Omega)$, there exists a unique global strong solution for the problem (1), (2) or (1), (3) for all $\frac{n}{\mu-1}\leqslant p<\infty$.

2 The Proofs of Theorem 1.1 and Theorem 1.2

In this section, we shall give the proofs of Theorem 1.1 and Theorem 1.2 by using Banach fixed point theorem and some a priori estimates.

Proof of Theorem 1.1

Let $T>0$ and set $X(T)=C([0,T];D(M))$, the space $X(T)$ is a Banach space with the maximum norm $\|\cdot\|_{X(T)}=\sup\limits_{0\leqslant t\leqslant T}\|\cdot\|_{\mu,p}$.

First we define an operator $G:X(T)\to X(T)$ by

$$Gu=u_0+\int_0^t M^{-1}\Delta u\mathrm{d}s+\int_0^t M^{-1}(b,\nabla u)\mathrm{d}s+\int_0^t M^{-1}u^r(a,\nabla u)\mathrm{d}s.\tag{13}$$

It is easy to know that G is well defined by Lemma 1.1 Next for any initial value $u_0\in D(M)$, we consider the unit ball

$$B_T=\{u(t)\mid u(t)\in X(T);\|u(t)-u_0(x)\|_{X(T)}\leqslant 1\}\tag{14}$$

in $X(T)$. Now we prove that $GB_T\subset B_T$ for suitable T. For this purpose, we do some a priori estimates. Since M^{-1} is a bounded linear operator, there is a constant $C>0$ such that

$$\|M^{-1}u\|_{\mu,p}\leqslant C\|u\|_p,\ \forall u\in L_p(\Omega).\tag{15}$$

By the definition of the equivalent norm and $\mu\geqslant 2$, we have

$$\|M^{-1}\Delta u\|_{\mu,p}\leqslant C\|u\|_{2,p}\leqslant C\|u\|_{\mu,p}\leqslant C(1+\|u_0\|_{\mu,p});\tag{16}$$

$$\|M^{-1}(b,\nabla u)\|_{\mu,p}\leqslant C\|(b,\nabla u)\|_p\leqslant |b|C\|u\|_{\mu,p}\leqslant C(1+\|u_0\|_{\mu,p});\tag{17}$$

for all $u\in B_T$; On the other hand, Sobolev embeding theorem implies

$$\|u\|_\infty\leqslant C\|u\|_{\mu,p},\ \forall u\in W^{\mu,p}(\Omega);\tag{18}$$

So

$$\|u\|_\infty\leqslant C(1+\|u_0\|_{\mu,p}),\ \forall u\in B_T;\tag{19}$$

And we have

$$\begin{aligned}\|M^{-1}u^r(a,\nabla u)\|_{\mu,p}&\leqslant C\|\nu^r(a,\nabla u)\|_p\leqslant C\|u\|_\infty^r\,|a|\,\|\nabla u\|_p\\&\leqslant C(1+\|u_0\|_{\mu,p}),\ \forall u\in B_T;\end{aligned}\tag{20}$$

Combining (13), (16), (17), (20), we have

$$\begin{aligned}\|Gu-u_0\|_{\mu,p}&\leqslant\int_0^t\|M^{-1}\Delta u\|_{\mu,p}\mathrm{d}s+\int_0^t\|M^{-1}(b,\nabla u)\|_{\mu,p}\mathrm{d}s+\int_0^t\|M^{-1}u^r(a,\nabla u)\|_{\mu,p}\mathrm{d}s\\&\leqslant CT(1+\|u_0\|_{\mu,p}),\end{aligned}\tag{21}$$

therefor, $GB_T\subset B_T$ when T is taken suitably small.

At last, we shall prove that $G: B_T\to B_T$ is a contraction mapping for suitable T. For any $u,v\in B_T$, we have

$$\begin{aligned}\|Gu-Gv\|_{\mu,p}\leqslant&\int_0^t\|M^{-1}(\Delta u-\Delta v)\|_{\mu,p}\mathrm{d}s+\int_0^t\|M^{-1}((b,\nabla \nu)-(b,\nabla v))\|_{\mu,p}\mathrm{d}s+\\&\int_0^t\|M^{-1}(u^r(a,\nabla u)-v^r(a,\nabla v))\|_{\mu,p}\mathrm{d}s;\end{aligned}\tag{22}$$

By (15), we have

$$\|M^{-1}(\Delta u-\Delta v)\|_{\mu,p}\leqslant C\|\Delta u-\Delta v\|_p\leqslant C\|u-v\|_{\mu,p};\tag{23}$$

$$\begin{aligned}\|M^{-1}((b,\nabla u)-(b,\nabla v))\|_{\mu,p}&\leqslant C\|(b,\nabla u)-(b,\nabla v)\|_p\\&\leqslant C|b|\,\|\nabla u-\nabla v\|_p\leqslant C|b|\,\|u-v\|_{\mu,p};\end{aligned}\tag{24}$$

$$\begin{aligned}\|M^{-1}(u^r(a,\nabla u)-v^r(a,\nabla v))\|_{\mu,p}&\leqslant C\|u^r(a,\nabla u)-v^r(a,\nabla v)\|_p\\&\leqslant\frac{C|a|}{r+1}\|\nabla u^{r+1}-\nabla v^{r+1}\|_p\\&\leqslant C|a|\,\|\,|\theta u+(1-\theta)v|^r\,|u-v|\,\|_{1,p};\end{aligned}\tag{25}$$

for $u,v\in B_T$, then $\theta u+(1-\theta)v\in B_T$ where $0\leqslant\theta\leqslant1$, by (18) then

$$\|\theta u+(1-\theta)v\|_\infty\leqslant C(1+\|u_0\|_{\mu,p});$$

then

$$\|M^{-1}(u^r(a,\nabla u)-v^r(a,\nabla v))\|_{\mu,p}\leqslant |a|C\|u-v\|_{1,p}\leqslant C\|u-v\|_{\mu,p};\tag{26}$$

Using (23), (24), (26), we obtain

$$\|Gu-Gv\|_{\mu,p}\leqslant CT\|u-v\|_{\mu,p}.\tag{27}$$

Thus if we choose T suitably small, G becomes a contraction mapping from B_T into itself. Hence, by Banach fixed point theorem, G has a unique fixed point $u(t)\in B_T$. Thus we complete the proof of Theorem 1.1.

Proof of Theorem 1.2 Let $T>0$, we consider Banach space

$$Y(T)=C([0,T];\mathcal{D}(M)\cap W^{\frac{\mu}{2},q}(\Omega));\tag{28}$$

$$Z(T)=C([0,T];\mathcal{D}(M)\cap W^{\mu,q}(\Omega));\tag{29}$$

Under norms $\|\cdot\|_{Y(T)}$, $\|\cdot\|_{Z(T)}$, which are defined by

$$\|u(\cdot)\|_{Y(T)}=\sup_{0\leqslant t\leqslant T}(\|u\|_{\mu,p}+\|u\|_{\frac{\mu}{2},q}),\ \forall u\in Y(T);\tag{30}$$

$$\|u(\cdot)\|_{Z(T)}=\sup_{0\leqslant t\leqslant T}(\|u\|_{\mu,p}+\|u\|_{\mu,q}),\ \forall u\in Z(T);\tag{31}$$

respectively. Substituting $X(T)$ by $Y(T)$, $Z(T)$ respectively in the proof of Theorem 1.1, we easily obtain the proof of Theorem 1.2.

3 The Proofs of Theorem 1.3 and Theorem 1.4

In this section, we shall establish the global existence theorem for the strong solution to the problem (1),

(2) and (1), (3). For this purpose, we first introduce a lemma which play an essential role in the global existence theory.

Lemma 3.1 Assume $u_0 \in W^{\frac{\mu}{2},2}(\Omega) \cap D(M)$ then the strong solution $u(t) = u(x,t)$, for the problem (1), (2) and (1), (3) on $[0,T) \times \Omega$ satisfies

$$\|u\|_{\frac{\mu}{2},2} \leqslant \|u_0\|_{\frac{\mu}{2},2} e^{CT} = C(T), \ \forall t \in [0,T). \tag{32}$$

Proof Multiplying equation (1) by u and integrating over Ω, we get

$$\int_\Omega M u_t u \mathrm{d}x = \int_\Omega \Delta u u \mathrm{d}x + \int_\Omega (b, \nabla u) u \mathrm{d}x + \int_\Omega u^r (a, \nabla u) u \mathrm{d}x; \tag{33}$$

Note that

$$\int_\Omega M u_t u \mathrm{d}x = C\int \quad (\xi)\hat{u}(\xi)\mathrm{d}\xi = C\int m(\xi)\hat{u}_t(\xi)\hat{u}(\xi)\mathrm{d}\xi = \frac{1}{2}C\int \frac{\mathrm{d}}{\mathrm{d}t}|(m(\xi))^{\frac{1}{2}}\hat{u}(\xi)|^2 \mathrm{d}\xi$$
$$= \frac{1}{2}C\frac{\mathrm{d}}{\mathrm{d}t}\|(m(\xi))^{\frac{1}{2}}\hat{u}(\xi)\|_2^2 = C\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\frac{\mu}{2},2}^2; \tag{34}$$

By integration by parts and the divergence theorem, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\frac{\mu}{2},2}^2 \leqslant C\|u\|_{\frac{\mu}{2},2}^2, \tag{35}$$

that is

$$\|u\|_{\frac{\mu}{2},2}^2 \leqslant \|u_0\|_{\frac{\mu}{2},2}^2 + C\int_0^t \|u\|_{\frac{\mu}{2},2}^2 \mathrm{d}s; \tag{36}$$

By Gronwall inequality, we obtain (32).

Proof of Theorem 1.3 Let $u(t)$ be the associated strong solution obtained by Theorem 1.1, we assume that $u(t)$ is defined on the maximal interval $[0,T)$, and $T < \infty$. To obtain global existence, it is sufficient to prove

$$\sup_{0 \leqslant t < T} \|u(t)\|_{\mu,p} < \infty; \tag{37}$$

otherwise we easily conclude the contraction.

Case (I): $n < \mu$.

By Sobolev embedding theorem and Lemma 3.1, we get

$$\|u\|_\infty \leqslant \|u\|_{\frac{\mu}{2},2} \leqslant C(T) < \infty. \tag{38}$$

Applying the operator M to both sides of the identity (10) and taking L_p-norm, by (38) we get

$$\|Mu\|_p \leqslant \|Mu_0\|_p + \int_0^t \|\Delta u\|_p \mathrm{d}s + \int_0^t \|(b, \nabla u)\|_p \mathrm{d}s + \int_0^t \|u^r(a, \nabla u)\|_p \mathrm{d}s$$
$$\leqslant \|u_0\|_{\mu,p} + C\int_0^t \|u\|_{\mu,p}\mathrm{d}s, \tag{39}$$

that is

$$\|u\|_{\mu,p} \leqslant \|u_0\|_{\mu,p} + C\int_0^t \|u\|_{\mu,p}\mathrm{d}s; \tag{40}$$

By Gronwall inequality, we obtain

$$\|u\|_{\mu,p} \leqslant C(T) < \infty, \ \forall t \in [0,T). \tag{41}$$

Case (II): $n \geqslant \mu$.

First, we consider the case of $\frac{n}{\mu} \leqslant p < \frac{n}{\mu - 1}$. Applying the operator M to both sides of (10) and taking L_p-norm, we get

$$\|Mu\|_p \leqslant \|Mu_0\|_p + \int_0^t \|u\|_{\mu,p}\mathrm{d}s + |b|\int_0^t \|u\|_{\mu,p}\mathrm{d}s + \int_0^t \|u^r(a, \nabla u)\|_p \mathrm{d}s; \tag{42}$$

It suffices to prove (37) on the case of $r \in \left[\frac{2(\mu-1)}{n}, \frac{2(\mu-1)}{n-\mu}\right]$, otherwise for $r = 0$ or $r \in \left(0, \frac{2(\mu-1)}{n}\right)$ we have when $r = 0$,

$$\|u^r(a,\nabla u)\|_p = \|(a,\nabla u)\|_p \leqslant |a|\,\|u\|_{\mu,p}; \tag{43}$$

When $r \in \left(0,\frac{2(\mu-1)}{n}\right)$, by Young inequality, we obtain

$$|u|^r \leqslant \left(1-\frac{nr}{2(\mu-1)}\right) + \frac{nr}{2(\mu-1)}|u|^{\frac{2(\mu-1)}{n}}; \tag{44}$$

$$\|\,|u|^r(a,\nabla u)\|_p \leqslant \left(1-\frac{nr}{2(\mu-1)}\right)\|(a,\nabla u)\|_p + \frac{nr}{2(\mu-1)}\|\,|u|^{\frac{2(\mu-1)}{n}}(a,\nabla u)\|_p \tag{45}$$

Thus the case $r \in \left(0,\frac{2(\mu-1)}{n}\right)$ turn into the case $r \in \left[\frac{2(\mu-1)}{n},\frac{2(\mu-1)}{n-\mu}\right]$.

By Hölder inequality

$$\|u^r(a,\nabla u)\|_p \leqslant \|u\|^r_{\frac{rn}{\mu-1}}\|\nabla u\|_h, \tag{46}$$

in which $\frac{1}{p} = \frac{\mu-1}{n} + \frac{1}{h}$, Sobolev embedding theorem implies that

$$\|u\|_{\frac{rn}{\mu-1}} \leqslant C\|u\|_{\frac{\mu}{2},2} \leqslant C(T) < \infty; \tag{47}$$

$$\|\nabla u\|_h \leqslant C\|u\|_{\mu,p}; \tag{48}$$

This gives

$$\|u^r(a,\nabla u)\|_p \leqslant C\|u\|_{\mu,p}; \tag{49}$$

From (39) and (49) we have

$$\|u\|_{\mu,p} \leqslant \|u_0\|_{\mu,p} + C\int_0^t \|u\|_{\mu,p}\,ds; \tag{50}$$

By Gronwall inequality, we get

$$\|u\|_{\mu,p} \leqslant C(T) < \infty, \ \forall t \in [0,T). \tag{51}$$

Case (III): we consider the case: $\frac{n}{\mu-1} \leqslant p < \infty$, $|\Omega| < \infty$.

For any $\frac{n}{\mu} < q < \frac{n}{\mu-1}$, Sobolev embedding theorem implies that

$$W^{\mu,p}(\Omega) \qquad W^{\mu,q}(\Omega). \tag{52}$$

Thus for $u_0 \in D(M) \cap W^{\mu,q}(\Omega) \cap W^{\frac{\mu}{2},2}(\Omega)$, according to Theorem 1.2 and the process of the proof above, we know that

$$\|u\|_{\mu,q} \leqslant C(T) < \infty. \tag{53}$$

Since $W^{\mu,q}(\Omega) \qquad L_\infty(\Omega)$, we have

$$\|u^r(a,\nabla u)\|_p \leqslant C\|u\|^r_{\mu,q}|a|\,\|\nabla u\|_p \leqslant C\|u\|_{1,p} \leqslant C\|u\|_{\mu,p}. \tag{54}$$

Combining (42) and (54), we get (50). Hence, Gronwall inequality implies that (36). The proof of Theorem 1.3 is completed.

Proof of Theorem 1.4　First, we consider the case $p > \frac{n}{\mu-1}$.

Applying the operator $M^{\frac{\mu-1}{\mu}}$ to both sides of (10) and taking L_p-norm, we have

$$\begin{aligned}\|M^{\frac{\mu-1}{\mu}}u\|_p \leqslant{}& \|M^{\frac{\mu-1}{\mu}}u_0(x)\|_p + \int_0^t\|M^{-\frac{1}{\mu}}\Delta\nu\|_p\,ds + \int_0^t\|M^{-\frac{1}{\mu}}(b,\nabla u)\|_p\,ds + \\ &\int_0^t\|M^{-\frac{1}{\mu}}u^r(a,\nabla u)\|_p\,ds,\end{aligned} \tag{55}$$

where $(\xi) = (m(\xi))^{\frac{\mu-1}{\mu}}\hat{u}(\xi)$. Notice that the operator $M^{-\frac{1}{\mu}}: L_p(\Omega) \to D(M^{\frac{1}{\mu}})$ is bounded linear and $\mu \geqslant 2$,

$$\|M^{-\frac{1}{\mu}}\Delta u\|_p \leqslant C\|M^{-\frac{1}{\mu}}\nabla u\|_{1,p} \leqslant C\|\nabla u\|_p \leqslant C\|u\|_{\mu-1,p}\,; \tag{56}$$

$$\|M^{-\frac{1}{\mu}}(b,\nabla u)\|_p \leqslant \|M^{-\frac{1}{\mu}}(b,\nabla u)\|_{1,p} \leqslant C\|(b,\nabla u)\|_p \leqslant |b|C\|u\|_{1,p} \leqslant C\|u\|_{\mu-1,p}. \tag{57}$$

By Sobolev embedding theorem, we have

$$\|M^{-\frac{1}{\mu}}u^r(a,\nabla u)\|_p \leqslant \|M^{-\frac{1}{\mu}}(u^r(a,\nabla u))\|_{1,p_1} \leqslant C\|u^r(a,\nabla u)\|_{p_1}, \tag{58}$$

in which $\frac{1}{p_1}=\frac{1}{n}+\frac{1}{p}$. By Hölder inequality,

$$\|u^r(a,\nabla u)\|_{p_1} \leqslant \|u\|_{nr}^r\|\nabla u\|_p. \tag{59}$$

From the condition of r and Sobolev embedding theorem imply

$$\|u\|_{nr} \leqslant \|u\|_{\frac{\mu}{2},2} \leqslant C(T)<\infty\,; \tag{60}$$

From (55) ~ (60), we obtain

$$\|u\|_{\mu-1,p} \leqslant \|u_0\|_{\mu-1,2}+C\int_0^t\|u\|_{\mu-1,p}\mathrm{d}s\,; \tag{61}$$

By Gronwall inequality

$$\|u\|_{\mu-1,p} \leqslant C(T)<\infty. \tag{62}$$

For $\frac{n}{\mu-1}<p$, Sobolev embedding theorem implies

$$\|u\|_\infty \leqslant C\|u\|_{\mu-1,p} \leqslant C(T)<\infty\,; \tag{63}$$

Now applying the operator M to both sides of (10) and taking L_p-norm, we get

$$\|Mu\|_p \leqslant \|Mu_0\|_p+\int_0^t\|\Delta u\|_p\mathrm{d}s+\int_0^t\|(b,\nabla u)\|_p\mathrm{d}s+\int_0^t\|u^r(a,\nabla u\|_p\mathrm{d}s\,; \tag{64}$$

From (63) and (64), we get

$$\|u\|_{\mu,p} \leqslant \|u_0\|_{\mu,p}+C\int_0^t\|u\|_{\mu,p}\mathrm{d}s\,; \tag{65}$$

By Gronwall inequality, we have

$$\|u\|_{\mu,p} \leqslant C(T)<\infty,\ \forall t\in[0,T). \tag{66}$$

Next, we consider the case of $p=\frac{n}{\mu-1}$.

For any $\frac{n}{\mu-1}=p<q\leqslant\frac{n}{\mu-2}$, Sobolev embedding theorem implies $W^{\mu,p}(\Omega)$ $W^{\mu-1,q}(\Omega)$, thus we have $u_0(x)\in W^{\frac{\mu}{2},2}(\Omega)\cap D(M)\cap W^{\mu-1,q}(\Omega)$. Applying $M^{\frac{\mu-1}{\mu}}$ to both sides of (10) and taking L_q-norm, just like the discussion of (62), we have

$$\|u\|_{\mu-1,q} \leqslant C(T)<\infty,\ \forall t\in[0,T)\,; \tag{67}$$

From Sobolev embedding theorem and (67), we obtain

$$\|u\|_\infty \leqslant \|u\|_{\mu-1,q} \leqslant C(T)<\infty,\ \forall t\in[0,T)\,; \tag{68}$$

Similar to the process of the proof in the first point, we also have (37). Thus Theorem 1.4 is valid.

Remark 3.1 If the terms $(b,\nabla u)+u^r(a,\nabla u)$ are replaced by more generalize form $\mathrm{div}\phi(u)$, which satisfies the condition in [13,14]

$$\phi(u)\in C^2(\mathbf{R},\mathbf{R}^n),\ |\phi(u)|\leqslant C(1+|u|^r)\,; \tag{69}$$

we can get the similar result as Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4.

Remark 3.2 Similar to the discussion before, we can study the inhomogeneous GBBM Burgers equation of higher order

$$Mu_t=\Delta u+\mathrm{div}\phi(u)+f(u) \tag{70}$$

with the IBV condition (2) and Cauchy condition (3), where $\phi(u)$ and $f(u)$ satisfies the conditions in [13, 14]. And we also have the similar results.

[References]

[1] Adams R A. Sobolev Spaces[M]. New York: Academic Press, 1975.

[2] Avrin J. The Generalized BBM Equation in $\mathbf{R}^n$ with Singular Initial Data[J]. Nonlinear Analysis T M A, 1987, 11(1): 139—147.

[3] Avrin J, Goldtein J A. Global Existence for the BBM Equation in Arbitrary Dimensions[J]. Nonlinear Analysis T M A, 1985, 9(8): 861—865.

[4] Benjamin T B, Bona J L, Mahony J J. Model Equation Long Waves in Nonlinear Dispersive Systems[J]. Phil Trans Soc Ser A, 1972, 272: 47—87.

[5] Biler P, Dziubański J, Hebisch W. Scattering of Small Solutions to Generalized Benjamin-Bona-Mahony Equation in Several Space dimensions[J]. Commun in Partial Differential Equations, 1992, 17(9—10): 1737—1758.

[6] Bona J L, Bryant P L. A Mathematical Model for Long Wave Generated by Wavemake in Dispersive Systems[J]. Pro Phil Soc, 1973, 73: 391—405.

[7] Dziubański J, Karch G. Nonlinear Scattering for Some Dispersive Equations Gereralizing Benjamin-Bona-Mahony Equation [J]. Monatshefte für, 1996, 122: 35—43.

[8] Gilbarg D, Trudinger N S. Elliptic Partial Differential Equations of Second Order[M]. Berlin, Heidelberg, New York: Springer-Verlag Press, 1998.

[9] Calvert B. The Equation $A(t,u'(t))+B(t,u(t))=0$[J]. Math Proc Phil Soc, 1976, 79: 545—561.

[10] Goldtein J A, Kajikiya R, Oharu S. On Some Nonlinear Dispersive Equation in Several Space Variable[J]. Differential and Integral Equations, 1990, 3(4): 617—632.

[11] Goldtein J A, Wichnoski B J. On the BBM Equation in Higher Dimensions[J]. Nonlinear Analysis T M A, 1980, 4(4): 665—675.

[12] Guo B. IBV Problem for One of Class of Multidimensional Inhomogeneous GBBM Equations[J]. Chin Ann Math Ser B, 1986, 8: 226—236.

[13] Guo B, Miao C. On Inhomogeneous GBBM Equation[J]. J Partial Diff Eqn, 1995, 18(3): 193—204.

[14] Miao C. On Inhomogeneous GBBM Equations of Higher Order[J]. Acta Mathematicae Applicatae Sinica, 1995, 18(4): 478—498.

[15] Miao C. Harmonic Analysis and Applications to Partial Differential Equations[M]. Beijing: Science Press, 1999.

[16] Stein E M. Harmonic Analysis: Real—Variable methods, Orthogonality, and Oscillatory Integrals[M]. Princeton, New Jersey: Princeton University Press, 1993.

[17] Stein E M. Singular Integral and Differential Property of Functions[M]. Princeton N J: Princeton University Press, 1970.

[18] Taylor M. Pseudodifferential operators[M]. Princeton, N J: Princeton University Press, 1981.

[19] Yunmei C. Remark on the Global Existence for the BBM Equation in Arbitrary Dimensions[J]. Applicable Analysis, 1988, 30(1): 1—15.

[责任编辑:陆炳新]