Representation of a Power As a Sum of Consecutive Factorials

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Abstract: Consider the equation $a^m = n! + (n+1)! + \cdots + (n+k)!$ with $a>1$, $m>1$, $n\geq 1$. We show that when $a \not\equiv 0 \bmod 223092870$, all solutions of the equation are $2^3 = 2! + 3!$, $3^2 = 1! + 2! + 3!$, $2^5 = 2! + 3! + 4!$ and $12^2 = 4! + 5!$; when $a \equiv 0 \bmod 223092870$, let p be the least prime with $p \nmid a$, if the equation has solutions, then $m \leq p$. Moreover, we conjecture that the foregoing four solutions are only solutions of the equation.

Key words: congruence, factorial, prime

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表方幂为连续的阶乘和

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[摘要] 考虑方程 $a^m = n! + (n+1)! + \cdots\cdots + (n+k)!$,其中 $a>1$, $m>1$, $n\geq 1$. 我们证明了当 $a \not\equiv 0 \bmod 223092870$ 时, 方程所有的解是 $2^3 = 2! + 3!$, $3^2 = 1! + 2! + 3!$, $2^5 = 2! + 3! + 4!$, $12^2 = 4! + 5!$; 当 $a \equiv 0 \bmod 223092870$ 时, 令 p 是满足 $p \nmid a$ 的最小素数,如果方程有解,则 $m \leq p$. 而且,我们猜想上述的四个解是方程仅有的解.

[关键词] 同余,阶乘,素数

0 Introduction

Erdös and Burr [2] ever conjectured that the largest solution of the equation

$$2^m = n_1! + n_2! + \cdots + n_k!, \text{ where } n_1 < n_2 < \cdots < n_k \tag{1}$$

is

$$2^7 = 2! + 3! + 5!.$$

Shen Lin ever proved this conjecture and found all solutions of Eq. (1) when the power of 2 is replaced by a power of 3. Grossman and Luca [3] investigated Eq. (1) when 2^m is replaced by a member of a given non-degenerate binary recurrence sequence. For other related problems, one can refer to [1][5].

Let a be a given positive integer with $a>1$. The following problem is natural and interesting: are there infinitely many m for which a^m can be represented as a sum of consecutive factorials?

In this paper, we investigate the equation

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$$a^m = n! + (n+1)! + \cdots + (n+k)!, \text{ where } a > 1,\ m > 1,\ n \geqslant 1 \qquad (2)$$

and show that the answer to the question is negative. In fact, the following results are proved.

Theorem 1 For $a \not\equiv 0 \bmod 223092870$, all solutions of Eq. (2) are

$$2^3 = 2! + 3!, 3^2 = 1! + 2! + 3!, 2^5 = 2! + 3! + 4! \text{ and } 12^2 = 4! + 5!.$$

Theorem 2 For $a \equiv 0 \bmod 223092870$, let p be the least prime with $p \nmid a$, if Eq. (2) has solutions, then $m \leqslant p$.

By Bertrand's postulate [4], it is easy to show that if $n > 1$, then $n!$ is not a power. Thus we need to consider the equation when $k \geqslant 1$. In the following, we always assume that $a > 1$, $k \geqslant 1$ and $m > 1$.

For convenience, we give the following notations: let $a \in \mathbf{Z}$ and p be a prime, if r is a nonnegative integer with $p^r \parallel a$ (that is, $p^r \parallel a$ but $p^{r+1} \nmid a$), then r is denoted by $ord_p(a)$. Let $L(n,k) = n! + (n+1)! + \cdots + (n+k)!$.

1 Proof of Theorem 1

Lemma 1 Let $a \in \mathbf{Z}^+$ and p be a prime. If $n \geqslant 3$ and $n = 2p - l, l = 1,3,4,5$, then $a^m \neq L(n,k)$ $(k \geqslant 1)$.

proof It is easy to see that

$$L(n,1) = (n+2)n!,\ L(n,2) = (n+2)^2 n!,\ L(n,3) = (n+2)n!\ (n^2+5n+5),$$
$$L(n,4) = (n+2)n!\ (n^2(n+5) + 4(n+1)(n+5) - 3).$$

Case 1 $l = 1$. Then $p \parallel n!$, $p^2 \mid (n+t)!$ $(t \geqslant 1)$.
Hence, for $m > 1$, we have $a^m \neq L(n,k)$ $(k \geqslant 1)$.

Case 2 $l = 3$. Then, by $n \geqslant 3$ we have

$$p \parallel n!, p \parallel (n+2)n!, p \parallel (n+2)^2 n! \text{ and } p^2 \mid (n+t)!\ (t \geqslant 3).$$

Hence, for $m > 1$, we have $a^m \neq L(n,k)$ $(k \geqslant 1)$.

Case 3 $l = 4$. Note that $n^2 + 5n + 5 = (n+4)(n+1) + 1$, we have $p \nmid n^2 + 5n + 5$.
Thus, by $n \geqslant 3$ we have $p \parallel (n+2)n!, p \parallel (n+2)^2 n!, p \parallel (n+2)n!\ (n^2+5n+5)$,
and $p^2 \mid (n+t)!$ $(t \geqslant 4)$.
Hence, for $m > 1$, we have $a^m \neq L(n,k)$ $(k \geqslant 1)$.

Case 4 $l = 5$. By $n \geqslant 3$, we have $p \geqslant 5$. If $p = 5$, then $n = 2p - 5 = 5 = 2 \times 3 - 1$. This is Case 1. Now we assume that $p > 5$. By $n = 2p - 5$, we have $p \mid n + 5$. Then

$$p \nmid n^2(n+5) + 4(n+1)(n+5) - 3, p \nmid n(n+5) + 5.$$

Thus

$$p \parallel (n+2)n!, p \parallel (n+2)^2 n!, p \parallel (n+2)n!\ (n^2+5n+5),$$
$$p \parallel (n+2)n!\ (n^2(n+5) + 4(n+1)(n+5) - 3), \text{ and } p^2 \mid (n+t)!\ (t \geqslant 5).$$

Hence, for $m > 1$, we have $a^m \neq L(n,k)$ $(k \geqslant 1)$.

This completes the proof of Lemma 1.

Lemma 2 For $n = 1,2,4$, all solutions of Eq. (2) are

$$2^3 = 2! + 3!,\ 3^2 = 1! + 2! + 3!,\ 2^5 = 2! + 3! + 4! \text{ and } 12^2 = 4! + 5!.$$

Proof **Case 1** $n = 1$, it is easy to compute that if $k \leqslant 7$, then the only solution of Eq. (2) is $3^2 = 1! + 2! + 3!$.

By $L(1,8) = 3^2 \times 5\,137 + 2^7 \times 3^4 \times 5 \times 7$, if $a^m = L(1,k)$ $(k \geqslant 8)$, then

$$3^2 \parallel a^m,\ m = 2 \text{ and } a^2 \equiv 3 \bmod 5, \text{ a contradiction.}$$

Hence, $a^m = L(1,k)$ if and only if $k = 2$.

Case 2 $n = 2$, it is easy to compute that if $k \leqslant 5$, then the only solutions of Eq. (2) are $2^3 = 2! + 3!$ and $2^5 = 2! + 3! + 4!$.

By $L(2,6) = 2^3 \times 739 + 2^7 \times 3^2 \times 5 \times 7$, if $a^m = L(2,k)$ $(k \geqslant 6)$, then

$$2^3 \parallel a^m,\ m = 3 \text{ and } a^3 \equiv 4 \bmod 7,$$

which contradicts with the fact that $a^3 \equiv 0, 1$ or $6 \bmod 7$ for $a \in \mathbf{Z}$.

Hence, $a^m = L(2,k)$ if and only if $k = 1$ or 2.

Case 3 $n = 4$, it is easy to compute that if $k \leqslant 3$, then the only solution of Eq. (2) is $12^2 = 4! + 5!$.

By $L(4,4) = 2^4 \times 3^2 \times 41 + 2^7 \times 3^2 \times 5 \times 7$, if $a^m = L(4,k)$ $(k \geqslant 4)$, then

$$2^4 \parallel a^m,\ m = 2, a^2 \equiv 3 \bmod 7 \text{ or } m = 4, a^4 \equiv 3 \bmod 7,$$

both contradict with the fact that $\left(\frac{3}{7}\right) = -1$.

Hence, $a^m = L(4,k)$ if and only if $k = 1$.

This completes the proof of Lemma 2.

Lemma 3 For $n = 8, 12, 14, 15, 16, 20$, Eq. (2) has no solutions.

Proof **Case 1** $n = 8$, it is easy to compute that $a^m \neq L(8,k)$ $(k \leqslant 3)$.

By $L(8,4) = 2^8 \times 3^2 \times 5^2 \times 7 \times 109 + 2^{10} \times 3^5 \times 5^2 \times 7 \times 11$, if $a^m = L(8,k)$ $(k \geqslant 4)$, then

$$3^2 \parallel a^m,\ m = 2, \text{ and } \frac{a^2}{2^8 \times 5^2 \times 3^2} \equiv 3 \bmod 4, \text{a contradiction.}$$

Hence, we have $a^m \neq L(8,k)$ $(k \geqslant 1)$.

Case 2 $n = 12$, it is easy to compute that $a^m \neq L(12,k)$ $(k \leqslant 2)$.

By $L(12,3) = 2^{12} \times 3^5 \times 5^2 \times 7^3 \times 11 + 2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13$, if $a^m = L(12,k)$ $(k \geqslant 3)$, then

$$3^5 \parallel a^m, m = 5 \text{ and } 5^2 \parallel a^m, m = 2, \text{a contradiction.}$$

Hence, we have $a^m \neq L(12,k)$ $(k \geqslant 1)$.

Case 3 $n = 14$, by $L(14,1) = 2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13 + 2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13$, if $a^m = L(14,k)$ $(k \geqslant 1)$, then

$$3^5 \parallel a^m,\ m = 5 \text{ and } 5^2 \parallel a^m,\ m = 2, \text{ a contradiction.}$$

Hence, we have $a^m \neq (14,k)$ $(k \geqslant 1)$.

Case 4 $n = 15$, it is easy to compute that $a^m \neq L(15,k)$ $(k \geqslant 2)$.

By $L(15,3) = 2^{11} \times 3^6 \times 5^3 \times 7^2 \times 11 \times 13 \times 17^2 + 2^{16} \times 3^8 \times 5^3 \times 7^2 \times 11 \times 13 \times 17$, if $a^m = L(15,k)$ $(k \geqslant 3)$, then

$$2^{11} \parallel a^m,\ m = 11 \text{ and } 3^6 \parallel a^m,\ m = 2, 3 \text{ or } 6, \text{ a contradiction.}$$

Hence, we have $a^m \neq L(15,k)$ $(k \geqslant 1)$.

Case 5 $n = 16$, it is easy to compute that $a^m \neq L(16,k)$ $(k \leqslant 3)$.

By $L(16,4) = 2^{16} \times 3^8 \times 5^3 \times 7^2 \times 11 \times 13 \times 341 + 2^{18} \times 3^8 \times 5^4 \times 7^2 \times 11 \times 13 \times 17 \times 19$, if $a^m = L(16,k)$ $(k \geqslant 4)$, then

$$5^3 \parallel a^m,\ m = 3 \text{ and } 2^{16} \parallel a^m,\ m = 2, 4, 8 \text{ or } 16, \text{ a contradiction.}$$

Hence, we have $a^m = L(16,k)$ $(k \geqslant 1)$.

Case 6 $n = 20$, we have

$L(20,1) = 2^{18} \times 3^8 \times 5^4 \times 7^2 \times 11 \times 13 \times 17 \times 19 + 2^{18} \times 3^9 \times 5^4 \times 7^3 \times 11 \times 13 \times 17 \times 19$, if $a^m = L(20,k)$ $(k \geqslant 1)$, then $7^2 \parallel a^m, m = 2$ and

$$\frac{a^2}{2^{18} \times 3^8 \times 5^4 \times 7^2} \equiv 3 \bmod 7,$$

which contradicts with the fact that $\left(\frac{3}{7}\right) = -1$.

Hence, we have $a^m \neq L(20,k)$ $(k \geqslant 1)$. This completes the proof of Lemma 3.

Proof of Theorem 1 By Lemma 1 and Lemma 3, we know that if $4 < n \leqslant 23$, then Eq. (2) has no solutions. Thus, if Eq. (2) has solutions, then $24! \mid a^m$ or $n \leqslant 4$.

If $24! \mid a^m$, then $a \equiv 0 \bmod 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23$,

that is

$$a \equiv 0 \bmod 223092870.$$

If $n \leqslant 4$, by Lemma 1 and Lemma 2, we know all solutions of Eq. (2) are

$$2^3 = 2! + 3!,\ 3^2 = 1! + 2! + 3!,\ 2^5 = 2! + 3! + 4! \text{ and } 12^2 = 4! + 5!.$$

Hence, when $a \not\equiv 0 \bmod 223092870$, all solutions of Eq. (2) are

$$2^3 = 2! + 3!,\ 3^2 = 1! + 2! + 3!,\ 2^5 = 2! + 3! + 4! \text{ and } 12^2 = 4! + 5!.$$

This completes the proof of Theorem 1.

2 Proof of Theorem 2

By the proof of Theorem 1, if Eq. (2) has solutions and $a \equiv 0 \bmod 223092870$, then $n \geqslant 24$. In the following, we assume that $n \geqslant 24$.

By p being the least prime with $p \nmid a$, we have $n \leqslant p-1$ and $p \geqslant 29$.

Case 1 $2 \mid n+1$.

If $a^m = n! + (n+1)! + \cdots + (n+k)!$ $(k \geqslant 1)$, then $2^{ord_2(n!)} \parallel a^m$.

Thus $m \leqslant ord_2(n!) < n \leqslant p-1$.

Case 2 $2 \nmid n+1$.

By Bertrand's postulate, there exists a prime q such that $\frac{n}{2} < q < n$. Then $2q > n$.

If $a^m = n! + (n+1)!$, then $a^m = (n+2)n!$. By $q^4 \nmid (n+2)n!$, we have $m \leqslant 3 < p$.

If $a^m = n! + (n+1)! + (n+2)!$, then $a^m = (n+2)^2 n!$. By $q^7 \nmid (n+2)^2 n!$, we have $m \leqslant 6 < p$.

If $a^m = n! + (n+1)! + (n+2)! + (n+3)!$, then $a^m = (n+2)n!\ (n^2+5n+5)$.

By $q^3 > \frac{n^3}{8} \geqslant 3n^2 > n^2+5n+5$, we have $q^3 \nmid n^2+5n+5$. Thus $q^7 \nmid (n+2)^2 n!\ (n^2+5n+5)$.

Hence, $m \leqslant 6 < p$.

Now we assume that $k \geqslant 4$.

Let $2^\alpha \parallel (n+2)(n^2+5n+5)n!$. By $2 \nmid n^2+5n+5$, we have $2^\alpha \parallel (n+2)n!$.

By $2 \nmid n+1$, we have $2^{\alpha+1} \mid (n+t)!$ $(t \geqslant 4)$.

Thus, if $a^m = (n+2)(n^2+5n+5)n! + \cdots + (n+k)!$, then $2^\alpha \parallel a^m$.

Hence $m \leqslant \alpha = ord_2((n+2)!) \leqslant n+1 \leqslant p$.

This completes the proof of Theorem 2.

Remark Up to now, we have verified that when $24 \leqslant n \leqslant 50$, Eq. (2) has no solutions. Thus, if Eq. (2) has other solutions different from the foregoing four solutions, then $a \equiv 0 \bmod 614889782588491410$. Hence, we have enough reasons to support the following conjecture.

Conjecture All solutions of Eq. (2) are

$$2^3 = 2! + 3!,\ 3^2 = 1! + 2! + 3!,$$
$$2^5 = 2! + 3! + 4!,\ 12^2 = 4! + 5!.$$

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