Disjoint Quasi-Kernels in Digraphs

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Abstract: A vertex set X of a digraph $D=(V,A)$ is a kernel if X is independent and for every $v\in V-X$ there exists $x\in X$ such that $vx\in A$. A vertex set X of a digraph $D=(V,A)$ is a quasi-kernel if X is independent and for every $v\in V-X$ there exist $w\in V-X$, $x\in X$ such that either $vx\in A$ or vw, $wx\in A$. In this paper, we provide a necessary condition and several sufficient conditions for a digraph to have a pair of disjoint quasi-kernels.

Key words: kernels, quasi-kernels, sink, digraphs

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有向图中不相交的准核

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[**摘要**] 有向图 D 的一个顶点集 X 被称为 D 的一个核,如果 X 是一个独立集并且 X 之外的每一个点都能经一步到达 X. 有向图 D 的一个顶点集 X 被称为 D 的一个准核,如果 X 是一个独立集并且 X 之外的每一个点都能经一步或两步到达 X. 在这篇文章中,我们给出了一个有向图有一对不相交的准核的一个必要条件和若干充分条件.

[**关键词**] 核, 准核,汇点,有向图

0 Introduction, Terminology and Notation

A vertex set X of a digraph $D=(V,A)$ is a kernel if X is independent and for every $v\in V-X$ there exists $x\in X$ such that $vx\in A$. A vertex set X of a digraph $D=(V,A)$ is a quasi-kernel if X is independent and for every $v\in V-X$ there exist $w\in V-X$, $x\in X$ such that either $vx\in A$ or vw, $wx\in A$. A digraph $T=(V,A)$ is a tournament if for every pair x, y of distinct vertices in V, either $xy\in A$ or $yx\in A$, but not both. A vertex of out-degree zero is called a sink. A vertex in a tournament is called a king if every other vertex is reachable by a directed path of length at most two. A quasi-kernel of cardinality 1 is called a 2-serf of D. For a digraph D, we denote the vertex (arc) set by $V(D)$ $(A(D))$. Let x,y be two vertices in D, if $xy\in A(D)$, we say x dominates y, and y is dominated by x, and denote it by $x\to y$. The closed in-neighbourhood (closed out-neighbourhood) of a vertex x is defined as follows.

$$N_D^-[x]=\{x\}\cup\{y\in V(D):y\to x\}\quad (N_D^+[x]=\{x\}\cup\{y\in V(D):x\to y\})$$

If the digraph under consideration is clear from the context, then we'll omit the subscript D. We use the standard terminology and notation on digraphs as given in [1]. Let x, y be two vertices of D and Y be a subset of $V(D)$. If $(x,y)\neq\varnothing((x,y)=\varnothing)$, then x dominates y (x doesn't dominate y). If $(x,Y)\neq\varnothing((x,Y)=\varnothing)$, then there exists (doesn't exist) a vertex $y\in Y$ such that $x\to y$. Let D be a digraph with n vertices, we

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may assume $V(D)=\{1,2,3,\cdots,n\}$. Now we replace each vertex i of D by an independent set $H_i(1\leqslant i\leqslant n)$ and if i dominates j, then each vertex of H_i dominates all the vertices of H_j. We denote the new digraph by $D[H_1,H_2,\cdots,H_n]$.

In 1974, Chvátal and Lovász[2] proved that every digraph has a quasi-kernel. In 1996, Jacob and Meyniel[4] proved that if a digraph D has no kernel, then D contains at least three quasi-kernels. The Jacob-Meyniel theorem extends the result of Moon[5] that every tournament with no sink has at least three 2-serfs. In [3], they characterize digraphs with exactly one and two quasi-kernels and provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels. In this paper, we study the disjoint quasi-kernels in digraphs. We start with a problem.

Problem If a digraph has no sink, then has it a pair of disjoint quasi-kernels?

The minimal counterexample is D in Fig. 1.

Obviously, D has no sink. Let T_7 denote the maximum tournament in D having the property that for every pair x, y of vertices there exists a vertex z such that $x\rightarrow z$ and $y\rightarrow z$. Each quasi-kernel of D contains exactly one vertex in T_7, since T_7 is a tournament. Considering any two quasi-kernels Q_x and Q_y containing the vertices x and y, respectively, they are not disjoint because both of them have to contain z', where $x\rightarrow z$ and $y\rightarrow z$.

But the converse proposition of the problem is right. (see Lemma 1)

1 Preliminaries

Lemma 1 If a digraph D has a pair of disjoint quasi-kernels, then D has no sink.

Proof If D has a sink x, then every quasi-kernel in D must contain x, a contradiction.

Lemma 2[3] Let x be a vertex in a digraph D. If x is a non-sink, then D has a quasi-kernel not including x.

Theorem 1[5] Every tournament without vertices of indegree-zero has at least three kings.

Since the converse of a tournament is a tournament, the above theorem can be reformulated for 2-serfs.

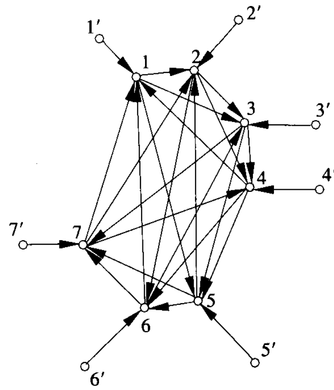

Fig 1.The minimal counterexample *D*.

Lemma 3 Every tournament with no sink has at least three 2-serfs.

Lemma 4[2] Every digraph has a quasi-kernel.

2 Main Results

Theorem 2 Let T be a tournament with n vertices. If T has no sink, then $T[H_1, H_2, \cdots, H_n]$ has at least three quasi-kernels and they are pairwisely disjoint.

Proof Let $V(T)=\{1,2,3,\cdots,n\}$. By Lemma 3, T has at least three 2-serfs. Let x, y, z be three 2-serfs of T. Hence, H_x, H_y, H_z are the quasi-kernels of $T[H_1,H_2,\cdots,H_n]$ and they are pairwisely disjoint.

Theorem 3 Let D be a digraph with no sink. If D has precisely two quasi-kernels, then they are disjoint.

Proof Let Q_1 and Q_2 be the only two quasi-kernels in D. If there exists a vertex $x\in Q_1\cap Q_2$, then it follows from Lemma 2 that x is a sink of D, a contradiction. Hence, Q_1 and Q_2 are disjoint.

Theorem 4 Let D be a digraph with no sink. If D possesses a quasi-kernel of cardinality at most two, then D has a pair of disjoint quasi-kernels.

Proof If D possesses a quasi-kernel Q_1 of cardinality 1, then we let $Q_1=\{x\}$. Since x is a non-sink, it follows from Lemma 2 that D has another quasi-kernel Q_2 not including x. Hence, Q_1 and Q_2 are disjoint quasi-kernels in D. If D possesses a quasi-kernel Q_1 of cardinality 2, then we let $Q_1=\{x,y\}$. Since y is a non-sink, there exists a vertex $z\in V(D)$ such that $y\rightarrow z$. Obviously, $z\neq x$. If $N^-[z]=V(D)$, then $Q_2=\{z\}$ is a quasi-kernel in D not containing x and y. Hence, Q_1 and Q_2 are disjoint quasi-kernels in D. If $N^-[z] \neq V(D)$, then we consider the following two cases.

Case 1 $x \in N^-(z) \setminus \{y\}$.

Let Q_2 be a quasi-kernel in $D - N^-[z]$. If z dominates a vertex in Q_2, then Q_2 is a quasi-kernel in D not containing x and y. If z doesn't dominate a vertex in Q_2, then $Q_3 = Q_2 \cup \{z\}$ is a quasi-kernel in D not containing x and y.

Case 2 $x \in D - N^-[z]$.

Subcase 2.1 x dominates a vertex in $D - N^-[z]$.

Clearly, x is a non-sink in $D - N^-[z]$. By Lemma 2, $D - N^-[z]$ has a quasi-kernel Q_2 not including x. If z dominates a vertex in Q_2, then Q_2 is a quasi-kernel in D not including x and y. If z doesn't dominate a vertex in Q_2, then $Q_3 = Q_2 \cup \{z\}$ is a quasi-kernel in D not including x and y.

Subcase 2.2 x doesn't dominate a vertex in $D - N^-[z]$.

Since x is a non-sink in D, x must dominate a vertex in $N^-(z) \setminus \{y\}$. Also, $N^+(x) \subseteq N^-(z) \setminus \{y\}$. Consider one vertex $w \in N^+(x)$. We delete $N^-(w)$ from $D - N^-[z]$. Let Q_2 be a quasi-kernel in $D - N^-[z] - N^-(w)$. If z doesn't dominate a vertex in Q_2, then $Q_3 = Q_2 \cup \{z\}$ is a quasi-kernel in D not including x and y. If z dominates a vertex in Q_2, then we consider the vertex w. If w doesn't dominate a vertex in Q_2, then $Q'_3 = Q_2 \cup \{w\}$ is a quasi-kernel in D not including x and y. If w dominates a vertex in Q_2, then Q_2 is a quasi-kernel in D not including x and y.

Therefore, D has a pair of disjoint quasi-kernels.

Theorem 5 Let D be a digraph with no sink or 2-cycle. If D possesses a quasi-kernel $Q_1 = \{x_1, x_2, x_3\}$ with $N^-(x_i) = \varnothing (i = 1, 2)$ and there's no triangle $w_1 w_3 w_2 w_1$ with $w_i \in N^+(x_i) (i = 1, 2, 3)$, then D has a pair of disjoint quasi-kernels.

Proof Since D has no sink, $N^+(x_i) \neq \varnothing (i = 1,2,3)$. If $N^+(x_1) \cap N^+(x_2) \neq \varnothing$, then we consider a vertex $w \in N^+(x_1) \cap N^+(x_2)$. Let Q be a quasi-kernel in $D - N^-[w]$. We need only to consider two cases: (1) $x_3 \in N^-(w) \setminus \{x_1, x_2\}$, (2) $x_3 \in D - N^-[w]$. By the same proof as Theorem 4, we can prove that D has a pair of disjoint quasi-kernels. If $N^+(x_1) \cap N^+(x_2) = \varnothing$, then we consider $w_1 \in N^+(x_1)$ and $w_2 \in N^+(x_2)$. Clearly, $w_1 \neq w_2$. We consider the following two cases.

Case 1 $x_3 \in N^-(w_1) \cup N^-(w_2)$.

Let Q be a quasi-kernel in $D - N^-[w_1] - N^-[w_2]$. Since D has no 2-cycle, we consider the following three subcases.

Subcase 1.1 $(w_1, w_2) = (w_2, w_1) = \varnothing$.

All the possible cases are listed in the following table:

$(w_1, Q) \neq \varnothing$	$(w_2, Q) \neq \varnothing$	$Q_2 = Q$
$(w_1, Q) \neq \varnothing$	$(w_2, Q) = \varnothing$	$Q_2 = Q \cup \{w_2\}$
$(w_1, Q) = \varnothing$	$(w_2, Q) \neq \varnothing$	$Q_2 = Q \cup \{w_1\}$
$(w_1, Q) = \varnothing$	$(w_2, Q) = \varnothing$	$Q_2 = Q \cup \{w_1, w_2\}$

Subcase 1.2 $(w_1, w_2) \neq \varnothing$ but $(w_2, w_1) = \varnothing$.

Subcase 1.3 $(w_1, w_2) = \varnothing$ but $(w_2, w_1) \neq \varnothing$.

Similarly as Subcase 1.1, we can get a quasi-kernel Q_2 in D.

Case 2 $x_3 \notin N^-[w_1] \cup N^-[w_2]$.

Clearly, $x_3 \in D' = D - N^-[w_1] - N^-[w_2]$. If x_3 dominates a vertex in D', then x_3 is a non-sink in D'. By Lemma 2, D' has a quasi-kernel Q not including x_3. Then we can similarly do as Case 1. If x_3 doesn't dominate a vertex in D', then $N^+(x_3) \subseteq N^-(w_1) \cup N^-(w_2)$. Since $N^-(x_i) = \varnothing (i = 1, 2)$, we have $N^+(x_3) \subseteq N^-(w_1) \cup N^-(w_2) \setminus \{x_1, x_2\}$. Without loss of generality, we may assume that x_3 dominates a vertex w_3 in

$N^-(w_2)\setminus\{x_2\}$. We delete $N^-(w_3)$ from D', and we get $D''=D-N^-[w_1]-N^-[w_2]-N^-(w_3)$. Let Q be a quasi-kernel in D''. Since D has no 2-cycle, we consider the following three subcases.

Subcase 2.1 $(w_1,w_2)=(w_2,w_1)=\varnothing$.

(1) $(w_1, Q)\neq\varnothing$, $(w_2, Q)\neq\varnothing$.

If $(w_3, Q)=\varnothing$, then $Q_2=Q\cup\{w_3\}$. Otherwise, $Q_2=Q$.

(2) $(w_1, Q)\neq\varnothing$, $(w_2, Q)=\varnothing$.

Whenever $(w_3, Q)=\varnothing$ or not, $Q_2=Q\cup\{w_2\}$.

(3) $(w_1, Q)=\varnothing$, $(w_2, Q)\neq\varnothing$.

If $(w_3, Q)\neq\varnothing$, then $Q_2=Q\cup\{w_1\}$. If $(w_3, Q)=\varnothing$ and $(w_1, w_3)=(w_3, w_1)=\varnothing$, then $Q_2=Q\cup\{w_1, w_3\}$. If $(w_3, Q)=\varnothing$ and $(w_1,w_3)\neq\varnothing$, then $Q_2=Q\cup\{w_3\}$. If $(w_3, Q)=\varnothing$ and $(w_3, w_1)\neq\varnothing$, then $Q_2=Q\cup\{w_1\}$.

(4) $(w_1, Q)=\varnothing$, $(w_2, Q)=\varnothing$.

Whenever $(w_3, Q)=\varnothing$ or not, $Q_2=Q\cup\{w_1, w_2\}$.

Subcase 2.2 $(w_1,w_2)=\varnothing$, $(w_2,w_1)\neq\varnothing$.

Subcase 2.3 $(w_1,w_2)\neq\varnothing$, $(w_2,w_1)=\varnothing$.

Similarly as Subcase 2.1, we get a quasi-kernel Q_2 in D.

Since D has no triangle $w_1w_3w_2w_1$ with $w_i\in N^+(x_i)(i=1, 2, 3)$, w_1 doesn't dominate w_3. Hence, either $(w_1,w_3)=(w_3,w_1)=\varnothing$ or $(w_3,w_1)\neq\varnothing$ but $(w_1,w_3)=\varnothing$.

(1) $(w_1, Q)\neq\varnothing$, $(w_2, Q)\neq\varnothing$.

If $(w_3, Q)=\varnothing$, then $Q_2=Q\cup\{w_3\}$. Otherwise, $Q_2=Q$.

(2) $(w_1, Q)\neq\varnothing$, $(w_2, Q)=\varnothing$.

Whenever $(w_3, Q)=\varnothing$ or not, $Q_2=Q\cup\{w_2\}$.

(3) $(w_1, Q)=\varnothing$, $(w_2, Q)\neq\varnothing$.

If $(w_3, Q)\neq\varnothing$, then $Q_2=Q\cup\{w_1\}$. If $(w_3, Q)=\varnothing$ and $(w_1, w_3)=(w_3, w_1)=\varnothing$, then $Q_2=Q\cup\{w_1, w_3\}$.

If $(w_3, Q)=\varnothing$ and $(w_3, w_1)\neq\varnothing$, then $Q_2=Q\cup\{w_1\}$.

(4) $(w_1, Q)=\varnothing$, $(w_2, Q)=\varnothing$.

If $(w_3, Q)\neq\varnothing$, then $Q_2=Q\cup\{w_1\}$. If $(w_3, Q)=\varnothing$ and $(w_1, w_3)=(w_3, w_1)=\varnothing$, then $Q_2=Q\cup\{w_1, w_3\}$.

If $(w_3, Q)=\varnothing$ and $(w_3, w_1)\neq\varnothing$, then $Q_2=Q\cup\{w_1\}$.

In Case 1 and Case 2, Q_2 is a quasi-kernel in D. Also, Q_1 and Q_2 are disjoint.

Problem Let D be a digraph with no sink. If D contains at least three quasi-kernels with each quasi-kernel of cardinality at least three, then what is a sufficient condition for D to have a pair of disjoint quasi-kernels (except the condition cited in Theorem 5)?

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