Hamiltonicity and the Independent Sets of Partially Square Graphs

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Abstract: The partially square graph G^* of G is a graph satisfying $V(G^*)=V(G)$ and $E(G^*)=E(G)\cup\{uv: uv\notin E(G)$, and $J(u,v)\neq\varnothing\}$. In this paper, we will use the technique of the vertex insertion on k or $(k+1)$-connected $(k\geqslant 2)$ graphs to provide a unified proof for G to be hamiltonian, 1-hamiltonian or hamiltonian-connected. The sufficient conditions are expressed by the inequality concerning $\sum_{i=1}^{k}|N(Y_i)|+b|N(y_0)|$ and $n(Y)$ in G for independent sets $Y=\{y_0,y_1,\cdots,y_k\}$ in G^*, where $b(0<b<k+1)$ is an integer, $Y_i=\{y_i,y_{i-1},\cdots,y_{i-(b-1)}\}\subseteq Y\backslash\{y_0\}$ for $i\in\{1,2,\cdots,k\}$ (the subscriptions of y'_js will be taken modulo k), and $n(Y)=|\{v\in V(G): \mathrm{dist}(v,Y)\leqslant 2\}|$.

Key words: hamiltonicity, vertex insertion, independent set, partially square graph

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哈密尔顿性和部分平方图的独立集

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[摘要] 设 G 是一个图,G 的部分平方图 G^* 满足 $V(G^*)=V(G)$,$E(G^*)=E(G)\cup\{uv:uv\in E(G)$,且 $J(u,v)\neq\varnothing\}$,这里 $J(u,v)=\{w\in N(u)\cap N(v),N(w)\subseteq N[u]\cup N[v]\}$. 本文利用插点方法,给出了关于 k,或 $(k+1)$-连通 $(k\geqslant 2)$ 图 G 是哈密尔顿的,1-哈密尔顿的或哈密尔顿连通的统一证明. 其充分条件是在图 G 中关于 $\sum_{i=1}^{k}|N(Y_i)|+b|N(y_0)|$ 与 $n(Y)$ 的不等式,这里 Y 是图 G 的部分平方图 G^* 的任一独立集,对于 $i\in\{1,2,\cdots,k\}$,$Y_i=\{y_i,y_{i-1},\cdots,y_{i-(b-1)}\}\subseteq Y(y_j$ 的下标将取模 $k)$;b 是一个整数,且 $0<b<k+1$;$n(Y)=|\{v\in V(G),\mathrm{dist}(v,Y)\leqslant 2\}|$.

[关键词] 哈密尔顿性,插点,独立集,部分平方图

0 Introduction

In this paper, the terminology and notation not defined will follow [1], and we consider simple finite graphs only. G will always stand for a graph. A cycle C of G is called a hamiltonian cycle if C is a spanning cycle, and a path P of G is called a hamiltonian path if P is a spanning path. A graph G is called hamiltonian if there exists a hamiltonian cycle in G. If $G-\{w\}$ is hamiltonian for any $w\in V(G)$, then G is 1-hamiltonian. G is called hamiltonian-connected graph if there exists a hamiltonian path in G which starts at u_1 and ends at u_2 for any $\{u_1, u_2\}\subseteq V(G)$.

Let G be a graph, for any $u\in V(G)$, let $N(u)$ denote the neighborhood of u and $d(u)=|N(u)|$ be the degree of u. For any $U\subseteq V(G)$, let $N(U)=\bigcup_{u\in U}N(u)$. Let U and R be subgraphs of G(or subsets of $V(G)$),

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Denote $N_R(U) = N(V) \cap R$.

Sufficient conditions involving degrees of vertices have been playing very important roles in studying the hamiltonicity of graphs.

Theorem 1[2] Let G be a graph of order $n \geqslant 3$. If the minimum degree $\delta \geqslant \frac{n}{2}$, then G is hamiltonian.

Theorem 2[3] Let G be a graph of order $n \geqslant 3$. If $d(u) + d(v) \geqslant n$ for every pair of nonadjacent vertices u and v, then G is hamiltonian.

Bondy improved and generalized Theorem 2 as follows.

Theorem 3[4] Let G be a k-connected graph of order $n \geqslant 3$. If $\sum_{i=1}^{k+1} d(z_i) > \frac{(k+1)(n-1)}{2}$ for every independent set $Z = \{z_1, z_2, \cdots, z_{k+1}\}$ of G, then G is hamiltonian.

We give the following notation.

Let $t > 1$ be an integer. Denote

$$I_t(G) = \{Y : Y \text{ is an independent set of } G, |Y| = t\}.$$

Let G be connected, $Y \subseteq V(G)$, and $v \in V(G)$. Denote $\operatorname{dist}(v, Y) = \min_{y \in Y} \{\operatorname{dist}(v, y)\}$ (where $\operatorname{dist}(v, y)$ stands for the distance between v and y),

$$N_i(Y) = \{v \in V(G) : \operatorname{dist}(v, Y) = i\} \quad (i = 0, 1, 2, \cdots), \text{ and}$$

$$n(Y) = |N_0(Y) \cup N_1(Y) \cup N_2(Y)| = |\{v \in V(G) : \operatorname{dist}(v, Y) \leqslant 2\}|.$$

Clearly, $N(Y) = N_1(Y)$, and $n(Y) \leqslant |V(G)|$. For each $i \in \{0, 1, 2, \cdots, |Y|\}$, denote

$$S_i(Y) = \{v \in V(G) : |N(v) \cap Y| = i\}.$$

For $v \in V(G)$, denote $N[v] = N(v) \cup \{v\}$. Let $\{u, v\} \subseteq V(G)$. Set

$$J(u, v) = \{w \in N(u) \cap N(v) : N(w) \subseteq N[u] \cup N[v]\}.$$

The partially square graph G^* [5] of G is a graph satisfying $V(G^*) = V(G)$ and $E(G^*) = E(G) \cup \{uv : uv \notin E(G), \text{ and } J(u, v) \neq \varnothing\}$.

In this paper, we will prove the following new results (Theorems 4 ~ 6) by using the vertex inserting lemmas introduced in [6]. In Theorems 4 ~ 6, we always assume that $Y = \{y_0, y_1, \cdots, y_k\} \in I_{k+1}(G)$,

$$Y_i = \{y_i, y_{i-1}, \cdots, y_{i-(b-1)}\} \subseteq Y \setminus \{y_0\}$$

for $i \in \{1, 2, \cdots, k\}$ (where the subscriptions of y'_js will be taken modulo k).

Theorem 4 Let G be a k-connected graph with $k \geqslant 2$, b an integer, and $0 < b < k+1$. If

$$\sigma_b(Y) = \sum_{i=1}^{k} |N(Y_i)| + b|N(y_0)| > \min\{k, \frac{2b-1+k}{2}\}(n(Y) - 1)$$

in G for each $Y \in I_{k+1}(G^*)$, then G is hamiltonian.

Theorem 5 Let G be a $(k+1)$-connected graph with $k \geqslant 2$, b an integer, and $0 < b < k+1$. If

$$\sigma_b(Y) = \sum_{i=1}^{k} |N(Y_i)| + b|N(y_0)| > \min\{k, \frac{2b-1+k}{2}\} n(Y)$$

in G for each $Y \in I_{k+1}(G^*)$, then G is 1-hamiltonian.

Theorem 6 Let G be a $(k+1)$-connected graph with $k \geqslant 3$, b an integer, and $0 < b < k+1$. If

$$\sigma_b(Y) = \sum_{i=1}^{k} |N(Y_i)| + b|N(y_0)| > \min\{k, \frac{2b-1+k}{2}\} n(Y)$$

in G for each $Y \in I_{k+1}(G^*)$, then G is hamiltonian-connected.

In Theorem 4, when $b = 1$, we have the following result.

Corollary Let G be a k-connected graph with $k \geqslant 2$. If $\sum_{i=0}^{k} |N(y_i)| > \frac{k+1}{2}(n(Y) - 1)$, in G for each $Y = \{y_0, y_1, \cdots, y_k\} \in I_{k+1}(G^*)$, then G is hamiltonian.

Clearly, Theorem 4 improves and generalizes Theorem 3, and Theorems 4 ~ 6 improve and generalize the corresponding theorems in [5], respectively.

Remark Let $G_k = K_{k+1,k}$ with $k \geqslant 2$ and bipartition (Y, U), where $|Y| = k+1$ and $|U| = k$. Clearly, G_k is a k-connected non-hamiltonian graph with $|V(G_k)| = 2k+1$. And we have $I_{k+1}(G_k^*) = I_{k+1}(G_k) = \{Y\}$, $n(Y) = |V(G_k)|$, and $N(y) = U$ for any $y \in Y$. Set $Y = \{y_0, y_1, \cdots, y_k\}$. Thus when $b = 1$,

$$\sigma_b(Y) = k(k+1) = \frac{1}{2}(k+1)(n(Y)-1) = \frac{2b-1+k}{2}(n(Y)-1).$$

Graph G_k shows that the lower bound of Theorem 4 can not be improved, and in this sense Theorem 4 is best possible.

1 The Basic Lemmas

In this section, we always assume that G is a connected non-hamiltonian graph and C is a maximal cycle of G (i. e., there is no cycle C' in G, such that $V(C) \subset V(C')$), and H is a component of $G - V(C)$. Assume also $\{v_1, v_2, \cdots, v_m\} \subseteq N_C(H)$ and $v_1, v_2, \cdots, v_m$ occur on C in the order of their indices. The subscriptions of v'_is will be taken modulo m. If $x \in V(C)$, denote by x^+ and x^- the successor and the predecessor of x along the orientation of C, respectively.

For each $i \in \{1, 2, \cdots, m\}$, a vertex $u \in C(v_i, v_{i+1})$ is called insertible[6] if there is some vertex $w \in C[v_{i+1}, v_i)$ such that $\{w, w^+\} \subseteq N(u)$. Otherwise u is called non-insertible.

Lemma 1[6] Let $u \in C(v_i, v_{i+1})$ for some $i \in \{1, 2, \cdots, m\}$. If all vertices in $C(v_i, u)$ are insertible, then $u \notin N_C(H)$. Therefore, there exists a vertex in $C(v_i, v_{i+1})$, which is non-insertible.

By Lemma 1, for each $i \in \{1, 2, \cdots, m\}$, let x_i be the first non-insertible vertex in $C(v_i, v_{i+1})$.

Let $X_M = \{x_0, x_1, \cdots, x_m\}$, where x_0 is an arbitrary vertex of H. Set $X \subseteq X_M$ such that $x_0 \in X$, and $|X| = k + 1 \leqslant m+1$. $X \backslash \{x_0\} = \{x_{p_1}, x_{p_2}, \cdots, x_{p_k}\}$ (where $1 \leqslant p_1 < p_2 < \cdots < p_k \leqslant m$). For convenience, we always assume that $x_{p_t} = x'_t$ and $v_{p_t} = v'_t$ for $t \in \{1, 2, \cdots, k\}$. Thus $X = \{x'_0, x'_1, \cdots, x'_k\}$, where $x'_0 = x_0$. Denote $J_X = \bigcup_{t=1}^{k} C[x'_t, v'_{t+1}]$, $K_X = V(G) \backslash J_X$.

The subscriptions of x'_is will be taken modulo k.

Lemma 2[6] $X_M \in I_{m+1}(G)$, $X \in I_{k+1}(G)$, $K_X \subseteq S_0(X) \cup S_1(X)$, $K_X \cap N_0(X) = \{x'_0\}$.

Lemma 3[5] $X_M \in I_{m+1}(G^*)$, Therefore $X \in I_{k+1}(G^*)$.

A segment $C[z_1, z_2)$ ($\subseteq C[x'_t, v'_{t+1}]$, $t \in \{1, 2, \cdots, k\}$) is called a CX-segment if

(i). $C(z_1, z_2) \cap S_0(X) = \varnothing$, and

(ii). $z_1 \in N_2(X) \cup X$, $z_2 \in S_0(X) \cup \{v'^{+}_{t+1}\}$.

A CX-segment $C[z_1, z_2)$ is said to be simple if $C(z_1, z_2) \subseteq S_1(X)$.

Lemma 4[7] Let $C[z_1, z_2)$ ($\subseteq C[x'_t, v'_{t+1}]$, $t \in \{1, 2, \cdots, k\}$) be a CX-segment. If $L_i = N(x'_i) \cap C(z_1, z_2)$ ($i \in \{0, 1, \cdots, k\}$), then

$$L_t, L_{t-1}, \cdots, L_1, L_k, L_{k-1}, \cdots, L_{t+1}, L_0$$

(some of them may be empty) form consecutive subpaths of $C(z_1, z_2)$ which can have only their endvertices in common, and $|L_i| \leqslant 1$ for $i \in \{0, 1, \cdots, k\} \backslash \{t\}$.

We always assume that b is an integer$(0 < b < k+1)$, and $b^* = \min\{k, \frac{2b-1+k}{2}\}$; $X_i = \{x'_i, x'_{i-1}, \cdots, x'_{i-(b-1)}\}$ ($\subseteq X \backslash \{x_0\}$) (for $i \in \{1, 2, \cdots, k\}$, and the subscriptions of x'_js will be taken modulo k).

Let $U \subseteq V(G)$. We always set

$$\sigma_b(U, X) = \sum_{i=1}^{k} |N(X_i) \cap U| + b|N(x_0) \cap U|;$$

$$\sigma_b(X) = \sigma_b(V(G), X) = \sum_{i=1}^{k} |N(X_i)| + b|N(x_0)|.$$

By the definition of $\sigma_b(X)$, it is not difficult to check that the following Lemma holds.

Lemma 5 (1) If $w \in S_1(X) \cap N(x'_q)$, then $\sigma_b(\{w\}, X) = b = b - 1 + |\{q\}|$.

(2) Let $w \in S_{i_0}(X) \cap C[x'_t, v'_{t+1}] \cap N(x'_{q_1}) \cap N(x'_{q_2}) \cap \cdots \cap N(x'_{q_{i_0}})$ $(i_0 \geqslant 2)$, where

$$(t \geqslant) q_1 > q_2 > \cdots > q_{i'_0} (>0)(k \geqslant) q_{i'_0+1} > \cdots > q_{i_0}.$$

then

$$\sigma_b(\{w\}, X) \leqslant \begin{cases} \min\{k, b - 1 + |\{q_1, q_1 - 1, q_1 - 2, \cdots, q_{i_0}\}|\} & \text{if } w \notin N(x_0) \\ \min\{k + b, 2b - 1 + |\{q_1, q_1 - 1, q_1 - 2, \cdots, q_{i_0} - 1\}|\} & \text{if } w \in N(x_0) \end{cases}$$

where the $q_1, q_1 - 1, q_1 - 2, \cdots$ will be taken modulo k and are not zero.

Lemma 6 (1) $\sigma_b(K_X, X) \leqslant b(|K_X| - 1 - |\bigcup_{l>2}(N_l(X) \cap K_X)|)$;

(2) If $C[z_1, z_2) \subseteq C[x'_t, v'_{t+1}]$ is a CX-segment, then $\sigma_b(C[z_1, z_2), X) \leqslant b^* |C[z_1, z_2)|$, and $\sigma_b(C[z_1, z_2), X) = b(|C[z_1, z_2)| - 1) < b^* |C[z_1, z_2)|$, when $C[z_1, z_2)$ is a simple CX-segment.

Proof Note that $b^* = \min\{k, \frac{2b - 1 + k}{2}\} \geqslant b$.

(1) By Lemma 2, $K_X \subseteq S_0(X) \cup S_1(X)$, and $K_X \cap N_0(X) = \{x'_0\}$. Thus by Lemma 5(1), we have

$$\begin{aligned} \sigma_b(K_X, X) &= b(|K_X| - |K_X \cap N_0(X)| - |\bigcup_{l \geqslant 2}(N_l(X) \cap K_X)|) \\ &\leqslant b(|K_X| - 1 - |\bigcup_{l>2}(N_l(X) \cap K_X)|), \end{aligned}$$

so (1) holds.

(2) By Lemma 4, we may assume $C(z_1, z_1'^{+}) = N(x'_t) \cap C[z_1, z_2)$, and $C(z_1, z'_1) \subseteq S_1(X)$. Let $W' = C(z_1, z'_1)$, $|W'| = h'$, $W = C[z'_1, z_2) = \{w_1, w_2, \cdots, w_h\}$, ($w_1, w_2, \cdots, w_h$ occur on C in the order of their indices), and $w_j \in S_{i_j}(X)$. Thus there exist $x'_{q_1^{(j)}}, x'_{q_2^{(j)}}, \cdots, x'_{q_{i_j}^{(j)}}$ $(i_j \geqslant 1)$ such that $w_j \in \bigcup_{t=1}^{i_j} N(x'_{q_t^{(j)}})$. By Lemma 4,

$$(t \geqslant) q_1^{(1)} > q_2^{(1)} > \cdots > q_{i_1}^{(1)} > q_1^{(2)} > q_2^{(2)} > \cdots > q_{i_2}^{(2)} > \cdots > q_1^{(j')} > q_2^{(j')} > \cdots > q_{i'_{j'}}^{(j')} (>0,$$
$$k \geqslant) q_{i'_{j'}+1}^{(j')} > \cdots > q_{i_{j'}}^{(j')} > \cdots > q_1^{(h)} > q_2^{(h)} > \cdots > q_{i_h}^{(h)},$$

$|C(z_1, z_2) \cap N(x_0)| \leqslant 1$, and $w_h \in N(x_0)$ if $|C(z_1, z_2) \cap N(x_0)| = 1$. Set $h_1 = |W \cap S_1(X)|$, and $h_2 = h - h_1$. Thus $|C[z_1, z_2)| = h' + h + 1 = h' + h_1 + h_2 + 1$, and $h_2 \leqslant k$.

Note that $C[z_1, z_2)$ is a simple CX-segment if and only if $h_2 = 0$. Thus if $C[z_1, z_2)$ is a simple CX-segment, then by Lemma 5(1),

$$\sigma_b(C[z_1, z_2), X) = b(h' + h_1) = b|C(z_1, z_2)| = b(|C[z_1, z_2)| - 1).$$

Therefore we may assume that $C[z_1, z_2)$ is not a simple CX-segment, so $h \neq 0$.

If $\frac{2b - 1 + k}{2} \geqslant k$, then by Lemma 5, it is not difficult to see that

$$\sigma_b(C[z_1, z_2), X) \leqslant kh_2 + b(h_1 + h') + b \leqslant k(h_2 + h_1 + h' + 1) = k|C[z_1, z_2)|.$$

If $\frac{2b - 1 + k}{2} < k$, then by Lemma 5, it is not difficult to see that

$$\begin{aligned} \sigma_b(C[z_1, z_2), X) &\leqslant (h(b - 1) + b + k) + bh' = bh' + (h + 1)(b - 1 + \frac{k + 1}{h + 1}) \\ &\leqslant \frac{2b - 1 + k}{2}(h + h' + 1) \leqslant \frac{2b - 1 + k}{2}|C[z_1, z_2)|, \end{aligned}$$

Thus (2) holds.

Lemma 7 $\sigma_b(X) \leqslant b^*(n(X) - 1)$.

Proof Consider $X = \{x'_0, x'_1, \cdots, x'_k\}$ and $J_X = \bigcup_{t=1}^{k} C[x'_t, v'_{t+1}]$. For $t \in \{1, 2, \cdots, k\}$, partition

$$C[x'_t, v'_{t+1}] \setminus \bigcup_{l>2}(N_l(X) \cap C[x'_t, v'_{t+1}]) \text{ into } s_t\ CX\text{-segments}$$
$$C[z_{11}^{(t)}, z_{12}^{(t)}), C[z_{21}^{(t)}, z_{22}^{(t)}), \cdots, C[z_{s_t1}^{(t)}, z_{s_t2}^{(t)}).$$

By Lemma 6, we have

$$\sigma_b(C[x'_t, v'_{t+1}], X) = \sum_{j=1}^{s_t} \sigma_b(C[z_{j1}^{(t)}, z_{j2}^{(t)}), X) \leqslant b^* \sum_{j=1}^{s_t} |C[z_{j1}^{(t)}, z_{j2}^{(t)})|$$

$$\leqslant b^*(|C[x'_t,v'_{t+1}]|-|\bigcup_{l>2}(N_l(X)\cap C[x'_t,v'_{t+1}])|).$$

Note that $J_X=\bigcup_{t=1}^{k}C[x'_t,v'_{t+1}]$. Thus

$$\sigma_b(J_X,X)=\sum_{t=1}^{k}\sigma_b(C[x'_t,v'_{t+1}],X)\leqslant\sum_{t=1}^{k}b^*(|C[x'_t,v'_{t+1}]|-|\bigcup_{l>2}(N_l(X)\cap C[x'_t,v'_{t+1}])|)$$
$$=b^*(|J_X|-|\bigcup_{l>2}(N_l(X)\cap J_X)|).$$

Note that $V(G)=J_X\cup K_X$, and $b\leqslant b^*$. So by Lemma 6(1), we have

$$\begin{aligned}\sigma_b(X)&=\sigma_b(J_X,X)+\sigma_b(K_X,X)\\&\leqslant b^*(|J_X|-|\bigcup_{l>2}(N_l(X)\cap J_X)|)+b(|K_X|-1-|\bigcup_{l>2}(N_l(X)\cap K_X)|)\\&\leqslant b^*(|J_X|-|\bigcup_{l>2}(N_l(X)\cap J_X)|+|K_X|-1-|\bigcup_{l>2}(N_l(X)\cap K_X)|)\\&=b^*(|V(G)\setminus\bigcup_{l>2}N_l(X)|-1)=b^*(n(X)-1).\end{aligned}$$

2 Proofs of the Theorems

Proof of Theorem 4 By reduction to absurdity. Suppose that G is non-hamiltonian. Since G is a k-connected graph with $k\geqslant 2$, we may choose a longest cycle C of G, a component H of $G-V(C)$, and $\{v_1,v_2,\cdots,v_k\}\subseteq N_C(H)$. Suppose that $v_1,v_2,\cdots,v_k$ occur on C in the order of their indices. By Lemma 1, for each $i\in\{1,2,\cdots,k\}$, choose x_i the first non-insertible vertex in $C(v_i,v_{i+1})$. Pick up an arbitrary $x_0\in V(H)$ and let $X=\{x_0,x_1,\cdots,x_k\}$, $X_i=\{x_i,x_{i-1},\cdots,x_{i-(b-1)}\}$ ($\subseteq X\setminus\{x_0\}$) (for $i\in\{1,2,\cdots,k\}$, and the subscriptions of x'_js will be taken modulo k). Lemma 7 shows that

$$\sigma_b(X)=\sum_{i=1}^{k}|N(X_i)|+b|N(x_0)|\leqslant b^*(n(X)-1).$$

On the other hand, Lemma 3 indicates that $X\in I_{k+1}(G^*)$, a contradiction.

The following lemma and the proofs of the following theorems will involve a graph G' other than G. In order to distinguish the notations such as $N(U)$, $n(X)$, $\sigma_b(X)$ introduced for G, we will simply add a prime to the notations with respect to G'. For example, $N'(U)$, $n'(X)$, $\sigma'_b(X)$ etc.

Proof of Theorem 5 By reduction to absurdity. Suppose that there exists some $w\in V(G)$ such that $G'=G-\{w\}$ is non-hamiltonian. Choose a cycle C of G' such that

(i) $|N'_C(w)|$ is maximum; (ii) subject to (i), C is maximal.

Let H be a component of $G'-V(C)$, and $N'_C(H)=\{v_1,v_2,\cdots,v_m\}$ with the convention that $v_1,v_2,\cdots,v_m$ occur on C in the order of their indices. Let x_i be the first non-insertible vertex in $C(v_i,v_{i+1})$ for each $i\in\{1,2,\cdots,m\}$. Let $X_M=\{x_0,x_1,\cdots,x_m\}$, where x_0 is an arbitrary vertex of H. By the proof of Theorem 9 in [7], there is $X\subseteq X_M$, such that $x_0\in X$ and $X\in I_{k+1}(G^*)$.

On the other hand, note that $b^*=\min\{k,\frac{2b-1+k}{2}\}\geqslant b$; and

$$\sigma_b(\{w\},X)=\sum_{i=1}^{k}|N(X_i)\cap\{w\}|+b|N(x_0)\cap\{w\}|\leqslant(k+b)\xi\leqslant 2b^*\xi,$$

where $\xi=0$ if $w\in S_0(X)$, otherwise $\xi=1$. Clearly, $n'(X)\leqslant n(X)-\xi$. Thus by Lemma 7, we have

$$\begin{aligned}\sigma_b(X)&=\sum_{i=1}^{k}|N(X_i)|+b|N(x_0)|\leqslant\sum_{i=1}^{k}|N'(X_i)|+b|N'(x_0)|+2b^*\xi\\&=\sigma'_b(X)+2b^*\xi\leqslant b^*(n'(X)-1)+2b^*\xi\leqslant b^*(n(X)-\xi-1+2\xi)\leqslant b^*n(X),\end{aligned}$$

a contradiction.

By the proof of Theorem 10 in [7], we have the following Lemma.

Lemma 8 Assume that G is a $(k+1)$-connected graph with $k\geqslant 3$, and there is some $\{u_1,u_2\}\subseteq V(G)$, G contains no (u_1,u_2)-hamiltonian-path. Assume that there exists a (u_1,u_2)-path P such that

(i) $V(P)\supseteq N(u_2)$;

(ii) subject to (i), $|N_P(u_1)|$ is maximum;

(iii) subject to (i), (ii), P is maximal.

Let H be a component of $G-V(P)$. Denote by G' the resulting graph obtained from G by adding a new vertex w and two new edges u_1w, u_2w. Then

(1) In G', $C=P[u_1,u_2]wu_1$ is a maximal cycle (choose the orientation of C that is the same as the orientation of P), but not hamiltonian cycle of G'. Let H be a component of $G'-V(C)$.

(2) Let $\{v_1,v_2,\cdots,v_m\}=N'_C(H)=N_P(H)$. Then $v_m\neq u_2$, and there exists the first non-insertible vertex x_i in $C(v_i,v_{i+1})$ for $i\in\{1,2,\cdots,m\}$, where $m\geqslant k+1\geqslant 4$; $X_M=\{x_0,x_1,\cdots,x_m\}\in I_{m+1}((G')^*)$, where x_0 is arbitrarily chosen in $V(H)$.

(3) There exists $X\subseteq X_M$, such that $x_0\in X$ and $X\in I_{k+1}(G^*)$.

Proof of Theorem 6 Suppose that graph G is not hamiltonian-connected. Then there is some $\{u_1,u_2\}\subseteq V(G)$, G contains no (u_1,u_2)-hamiltonian-path. By Theorem 5, there is a hamiltonian cycle C' in $G-u_2$. Choose an orientation of C'. Let $C'(u'_2,u_1)\cap N(u_2)=\varnothing$ and $u'_2\in N(u_2)$. The (u_1,u_2)-path $C'[u_1,u'_2]u_2$ contains the set $N(u_2)$. Thus one can choose a (u_1,u_2)-path P such that

(i) $V(P)\supseteq N(u_2)$;

(ii) subject to (i), $|N_P(u_1)|$ is maximum;

(iii) subject to (i), (ii), P is maximal.

Let H be a component of $G-V(P)$. Add a new vertex w and two new edges u_1w, u_2w to G and denote by G' the resulting graph. By Lemma 8(1), $C=P[u_1,u_2]wu_1$ is a maximal cycle in G' (choose the orientation of C agree with that of P), but not hamiltonian cycle of G'; H is a component of $G'-V(C)$. Let $\{v_1,v_2,\cdots,v_m\}=N'_C(H)=N_P(H)$. By Lemma 8(2), $v_m\neq u_2$, there exists the first non-insertible vertex x_i in $C(v_i,v_{i+1})$ for $i\in\{1,2,\cdots,m\}$, where $m\geqslant k+1\geqslant 4$. Let $X_M=\{x_0,x_1,\cdots,x_m\}\in I_{m+1}((G')^*)$, where x_0 is arbitrarily chosen in $V(H)$. By Lemma 8(3), there exists $X\subseteq X_M$, such that $x_0\in X$ and $X\in I_{k+1}(G^*)$.

On the other hand, by the construction of G', $n'(X)\leqslant n(X)+1$. Thus by Lemma 7, it is easy to see that

$$\begin{aligned}\sigma_b(X) &= \sum_{i=1}^{k}|N(X_i)|+b|N(x_0)|\leqslant\sum_{i=1}^{k}|N'(X_i)|+b|N'(x_0)|\\ &=\sigma'_b(X)\leqslant b^*(n'(X)-1)\leqslant b^*n(X),\end{aligned}$$

a contradiction.

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