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New Proof of a Result about the Coloring of Distance Graphs

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Abstract: Using the method of number theory, we redetermine the circular chromatic number $\chi_c(D)$ and fractional chromatic number $\chi_f(D)$ of the distance graph $G(Z,D)$, where $D=\{a,b,a+b,2(a+b)\}$ is a special 4-elements distance set.

Key words: distance graph, circular chromatic number, fractional chromatic number, star-extremal graph, Diophantine approximation

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关于距离图着色问题一个结果的新证明

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[摘要] 利用数论的方法,重新确定了距离图 $G(Z,D)$ 的圆色数 $\chi_c(D)$ 和分式色数 $\chi_f(D)$,其中 $D=\{a,b,a+b,2(a+b)\}$ 是一个特殊的四元素距离集.

[关键词] 距离图,圆色数,分式色数,星极图,丢番图逼近

0 Introduction

Let $G(Z,D)$ denote the graph with the set Z of integers as vertex set and with an edge joining two vertices u and v if and only if $|u-v|\in D$. Such a graph $G(Z,D)$ is called integer distance graph or simply distance graph (with a distance set D). Distance graphs, first studied by Eggleton et. al. [1], were motivated by the well-known plane-coloring problem: what is the minimum number of colors needed to color all points of a Euclidean plane so that points at unit distance are colored with different colors? This problem is equivalent to determining the chromatic number of the distance graph $G(R^2,\{1\})$. It is well known that the chromatic number of the distance graph is between 4 and 7. However the exact number of colors needed remains unknown.

Let k and d be positive integers such that $k\geqslant 2d$. A (k,d)-coloring of a graph $G=(V,E)$ is a mapping $c: V \to Z_k=\{0,1,2,\cdots,k-1\}$ such that $|c(u)-c(v)|_k\geqslant d$ for each edge $uv\in E$, where $|x|_k=\min\{|x|,k-|x|\}$. In fact, it is the generalization of an ordinary k-coloring of G which is just a $(k,1)$-coloring. The circular chromatic number, also called star-chromatic number, of G is defined as

$$\chi_c(G)=\inf\{k/d: G \text{ has a } (k,d)\text{-coloring}\}.$$

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For a real number x, denote by $\| x \|$ the distance from x to the nearest integer, let $\| tX \| = \inf\{\| tx \| : x \in X\}$, and let $\kappa(X) = \sup\{\| tX \| : t \in \mathbf{R}\}$, then we know that $\chi_c(G(Z,X)) \leqslant \frac{1}{\kappa(X)}$ (see [2]).

Another generalization of the ordinary coloring is the fractional coloring. A mapping c from the collection φ of independent sets of a graph G to the interval $[0,1]$ is a fractional-coloring if $\sum_{S \in \varphi, x \in S} c(S) \geqslant 1$ for every vertex x of G. The value of a fractional-coloring c is $\sum_{S \in \varphi} c(S)$. The fractional – chromatic number $\chi_f(G)$ is the minimum of the values of fractional colorings of G. It is clear from the definitions that $\chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G)$. For a graph G which satisfies $\chi_c(G) = \chi_f(G)$, we call it star-extremal.

The chromatic number $\chi(D)$ of the distance graph $G(Z,D)$ has been completely determined when $|D| \leqslant 3$. But there are a few, far from complete, results about the chromatic numbers of the distance graphs when $|D| \geqslant 4$. Some conclusions were given in [3,4,5]. For further related information see [6,7].

1 Main Theorem

In [8], using the method of graph theory, Xu and Song determined the circular chromatic number $\chi_c(D)$ and fractional chromatic number $\chi_f(D)$ of the distance graph $G(Z,D)$ for special 4-elements distance set $D = \{a, b, a+b, 2(a+b)\}$. In this paper, using the method of number theory, we give a new proof of this result:

Theorem (see [8]) Suppose $D = \{a, b, a+b, 2(a+b)\}$, $0 < a < b$, $\gcd(a,b) = 1$,

(i) If $b - a = 3k$, then $\chi_f(D) = \chi_c(D) = \chi(D) = 3$.

(ii) If $b - a = 3k+1$, then $\chi_f(D) = \chi_c(D) = 3 + 1/(a+k)$, $\chi(D) = 4$.

(iii) If $b - a = 3k+2$, then $\chi_f(D) = \chi_c(D) = 3 + 1/(a+2k+1)$, $\chi(D) = 4$.

Lemma 1 (see [9]) Let $0 < \lambda \leqslant \frac{1}{2}$ be a real number and $a_1, a_2, \cdots, a_n$ be n positive real numbers. The following statements are equivalent:

(A) There exist n integers $k_1, k_2, \cdots, k_n$ such that

$$a_i k_j - a_j k_i \leqslant (1-\lambda) a_j - \lambda a_i, \qquad i, \quad j = 1, 2, \cdots, n.$$

(B) There is a real number x such that each $\| a_i x \| \geqslant \lambda$, $i = 1, 2, \cdots, n$.

Lemma 2 [Theorem 1(c), 9] For any three positive integers $a_1 \leqslant a_2 \leqslant a_3$ such that $\gcd(a_1, a_2, a_3) = 1$. If $a_3 = a_1 + a_2$, then there exist three integers k_1, k_2, k_3 such that

$$a_i k_j - a_j k_i \leqslant (1-\lambda) a_j - \lambda a_i, \; i, j = 1, 2, 3,$$

where

$$\lambda = \begin{cases} \frac{1}{3}, & a_2 - a_1 = 3k; \\ \frac{1}{3} - \frac{1}{3} \frac{1}{a_1 + a_3}, & a_2 - a_1 = 3k+1; \\ \frac{1}{3} - \frac{1}{3} \frac{1}{a_2 + a_3}, & a_2 - a_1 = 3k+2. \end{cases}$$

Lemma 3 (see [10]) Suppose $D = \{a, b, a+b\}$, $0 < a < b$, $\gcd(a,b) = 1$.

(i) If $b - a = 3k$, then $\chi_f(D) = \chi_c(D) = \chi(D) = 3$.

(ii) If $b - a = 3k+1$, then $\chi_f(D) = \chi_c(D) = 3 + 1/(a+k)$, $\chi(D) = 4$.

(iii) If $b - a = 3k+2$, then $\chi_f(D) = \chi_c(D) = 3 + 1/(a+2k+1)$, $\chi(D) = 4$.

Lemma 4 Suppose a_1, a_2, a_3, a_4 are four positive integers such that $a_1 < a_2$, $a_3 = a_1 + a_2$, $a_4 = 2a_3$ and $\gcd(a_1, a_2) = 1$, then there exist four integers k_1, k_2, k_3, k_4 such that

$$a_i k_j - a_j k_i \leqslant (1-\lambda) a_j - \lambda a_i, \qquad i, \quad j = 1, 2, 3, 4,$$

where

$$\lambda = \begin{cases} \frac{1}{3}, & a_2 - a_1 = 3k; \\ \frac{1}{3} - \frac{1}{3}\frac{1}{a_1 + a_3}, & a_2 - a_1 = 3k+1; \\ \frac{1}{3} - \frac{1}{3}\frac{1}{a_2 + a_3}, & a_2 - a_1 = 3k+2. \end{cases}$$

Proof For any two integers m, n, let

$$A = m + \frac{a_1 - a_2}{2}, \ r = n - \frac{1}{2}.$$

Since $\gcd(a_1, a_2) = 1$, there exist two integers m_1, m_2 such that

$$a_1 m_2 - a_2 m_1 = m.$$

Let

$$k_3 = n + m_1 + m_2, \ k_1 = m_1, \ k_2 = m_2.$$

Then we have

$$A + a_1 r = a_1 k_3 - a_3 k_1 + \frac{a_1 - a_3}{2}, \tag{1}$$

$$-A + a_2 r = a_2 k_3 - a_3 k_2 + \frac{a_2 - a_3}{2}. \tag{2}$$

From the proof of Lemma 2, we know

Case 1. $a_2 - a_1 = 3k$. In this case, $A = \frac{k}{2}$ and $r = \frac{1}{2}$, there exist the corresponding three integers k_1, k_2, k_3 such that

$$a_i k_j - a_j k_i \leqslant (1-\lambda) a_j - \lambda a_i, \quad \lambda = \frac{1}{3}, \ i, j = 1, 2, 3.$$

Put $k_4 = 2k_3$, then by (1), (2), we have

$$a_1 k_4 - a_4 k_1 = 2(a_1 k_3 - a_3 k_1) = 2\left(A + a_1 r - \frac{a_1 - a_3}{2}\right) = \frac{4}{3}a_3 - \frac{2}{3}a_1,$$

$$a_2 k_4 - a_4 k_2 = 2(a_2 k_3 - a_3 k_2) = 2\left(-A + a_2 r - \frac{a_2 - a_3}{2}\right) = \frac{4}{3}a_3 - \frac{2}{3}a_2.$$

When $\lambda = \frac{1}{3}$, it is easy to obtain that

$$\lambda a_4 - (1-\lambda) a_1 = \frac{1}{3}a_4 - \frac{2}{3}a_1 < \frac{4}{3}a_3 - \frac{2}{3}a_1 < \frac{2}{3}a_4 - \frac{1}{3}a_1 = (1-\lambda) a_4 - \lambda a_1,$$

$$\lambda a_4 - (1-\lambda) a_2 = \frac{1}{3}a_4 - \frac{2}{3}a_2 < \frac{4}{3}a_3 - \frac{2}{3}a_2 < \frac{2}{3}a_4 - \frac{1}{3}a_2 = (1-\lambda) a_4 - \lambda a_2,$$

$$\lambda a_4 - (1-\lambda) a_3 = \frac{1}{3}a_4 - \frac{2}{3}a_3 = 0 = a_3 k_4 - a_4 k_3 < \frac{2}{3}a_4 - \frac{1}{3}a_3 = (1-\lambda) a_4 - \lambda a_3.$$

Thus, we have k_1, k_2, k_3, k_4 such that

$$a_i k_j - a_j k_i \leqslant (1-\lambda) a_j - \lambda a_i, \ \lambda = \frac{1}{3}, \ i, j = 1, 2, 3, 4.$$

Case 2. $a_2 - a_1 = 3k+1$. In this case, $A = \frac{k+1}{2}$ and $r = \frac{1}{2}$, there exist the corresponding three integers k_1, k_2, k_3 such that

$$a_i k_j - a_j k_i \leqslant (1-\lambda) a_j - \lambda a_i, \quad \lambda = \frac{1}{3} - \frac{1}{3}\frac{1}{a_1 + a_3}, \ i, j = 1, 2, 3.$$

Put $k_4 = 2k_3$, then by (1), (2), we have

$$a_1 k_4 - a_4 k_1 = 2(a_1 k_3 - a_3 k_1) = 2\left(A + a_1 r - \frac{a_1 - a_3}{2}\right) = \frac{4}{3}a_3 - \frac{2}{3}a_1 + \frac{2}{3},$$

$$a_2k_4 - a_4k_2 = 2(a_2k_3 - a_3k_2) = 2\left(-A + a_2r - \frac{a_2 - a_3}{2}\right) = \frac{4}{3}a_3 - \frac{2}{3}a_2 - \frac{2}{3}.$$

When $\lambda = \frac{1}{3} - \frac{1}{3}\frac{1}{a_1 + a_3}$, it is easy to obtain that

$$\lambda a_4 - (1-\lambda)a_1 < \frac{4}{3}a_3 - \frac{2}{3}a_1 + \frac{2}{3} < (1-\lambda)a_4 - \lambda a_1,$$

$$\lambda a_4 - (1-\lambda)a_2 < \frac{4}{3}a_3 - \frac{2}{3}a_2 - \frac{2}{3} < (1-\lambda)a_4 - \lambda a_2,$$

$$\lambda a_4 - (1-\lambda)a_3 < 0 = a_3k_4 - a_4k_3 < (1-\lambda)a_4 - \lambda a_3.$$

Thus, we have k_1, k_2, k_3, k_4 such that

$$a_ik_j - a_jk_i \leqslant (1-\lambda)a_j - \lambda a_i, \lambda = \frac{1}{3} - \frac{1}{3}\frac{1}{a_1 + a_3}, \ i, j = 1,2,3,4.$$

Case 3. $a_2 - a_1 = 3k + 2$. In this case, $A = \frac{k}{2}$ and $r = \frac{1}{2}$, there exist the corresponding three integers k_1, k_2, k_3 such that $a_ik_j - a_jk_i \leqslant (1-\lambda)a_j - \lambda a_i, \lambda = \frac{1}{3} - \frac{1}{3}\frac{1}{a_2 + a_3}$, $i, j = 1,2,3$.

Put $k_4 = 2k_3$, then by (1), (2), we have

$$a_1k_4 - a_4k_1 = 2(a_1k_3 - a_3k_1) = 2\left(A + a_1r - \frac{a_1 - a_3}{2}\right) = \frac{4}{3}a_3 - \frac{2}{3}a_1 - \frac{2}{3},$$

$$a_2k_4 - a_4k_2 = 2(a_2k_3 - a_3k_2) = 2\left(-A + a_2r - \frac{a_2 - a_3}{2}\right) = \frac{4}{3}a_3 - \frac{2}{3}a_2 + \frac{2}{3}.$$

When $\lambda = \frac{1}{3} - \frac{1}{3}\frac{1}{a_2 + a_3}$, it is easy to obtain that

$$\lambda a_4 - (1-\lambda)a_1 < \frac{4}{3}a_3 - \frac{2}{3}a_1 - \frac{2}{3} < (1-\lambda)a_4 - \lambda a_1,$$

$$\lambda a_4 - (1-\lambda)a_2 < \frac{4}{3}a_3 - \frac{2}{3}a_2 + \frac{2}{3} < (1-\lambda)a_4 - \lambda a_2,$$

$$\lambda a_4 - (1-\lambda)a_3 < 0 = a_3k_4 - a_4k_3 < (1-\lambda)a_4 - \lambda a_3.$$

Thus, we have k_1, k_2, k_3, k_4 such that

$$a_ik_j - a_jk_i \leqslant (1-\lambda)a_j - \lambda a_i, \lambda = \frac{1}{3} - \frac{1}{3}\frac{1}{a_2 + a_3}, \ i, j = 1,2,3,4.$$

This completes the proof of Lemma 4.

Remark If we replace the above condition $a_4 = 2a_3$ by $a_4 = na_3 (n \neq 2)$, then the result may not hold. For example, $a_1 = 1$, $a_2 = 4$, $a_3 = 5$, $a_4 = 15$. If the above result holds, then there exist four integers k_1, k_2, k_3, k_4 such that

$$a_ik_j - a_jk_i \leqslant (1-\lambda)a_j - \lambda a_i, \ \lambda = \frac{1}{3}, \ i, j = 1,2,3,4.$$

Thus we have

$$1 \leqslant k_3 - 5k_1 \leqslant 3, \ \frac{13}{3} \leqslant k_4 - 15k_1 \leqslant \frac{29}{3}, \quad \frac{7}{3} \leqslant 4k_4 - 15k_2 \leqslant \frac{26}{3}, \ \frac{5}{3} \leqslant 5k_4 - 15k_3 \leqslant \frac{25}{3}.$$

Thus,

$$k_3 = 5k_1 + 1, \ 5k_1 + 2, \ 5k_1 + 3; \tag{3}$$

$$k_4 = 15k_1 + 5, \ 15k_1 + 6, \cdots, \ 15k_1 + 9; \tag{4}$$

$$4k_4 - 15k_2 = 3, 4, \cdots, 8; \tag{5}$$

$$k_4 = 3k_3 + 1 = 15k_1 + 4, \ 15k_1 + 7, 15k_1 + 10. \tag{6}$$

By (4), (6), we have $k_4 = 15k_1 + 7$.

By (5), we have $15k_2 = 4k_4 - j = 4(15k_1 + 7) - j = 60k_1 + 28 - j$, $j = 3, \cdots, 8$. Thus, $15k_2 = 60k_1 + 25$, $60k_1 + 24, \cdots, 60k_1 + 20$, then $k_2 \notin \mathbf{Z}$, a contradiction.

Proof of Theorem Since the graph $G(Z, \{a,b,a+b\})$ is a subgraph of distance graph $G(Z, \{a,b,a+b,2(a+b)\})$, we have the inequality $\chi_f(\{a,b,a+b\}) \leqslant \chi_f(\{a,b,a+b,2(a+b)\})$. Thus, by Lemma 3 we have

$$\chi_f(\{a,b,a+b,2(a+b)\}) \geqslant \begin{cases} 3, & b-a=3k; \\ 3+1/(a+k), & b-a=3k+1; \\ 3+1/(a+2k+1), & b-a=3k+2. \end{cases} \tag{7}$$

By Lemma 1, we have $\kappa(D) \geqslant \lambda$. By Lemma 4 and the fact that $\chi_c(G(Z,D)) \leqslant \frac{1}{\kappa(D)}$ [2], we have

$$\chi_c(\{a,b,a+b,2(a+b)\}) \leqslant \begin{cases} 3, \ b-a=3k; \\ 3+1/(a+k), \ b-a=3k+1; \\ 3+1/(a+2k+1), \ b-a=3k+2. \end{cases} \tag{8}$$

Therefore, the chromatic numbers of the distance graph $G(Z,D)$ are given:

(i) If $b-a=3k$, then $\chi_f(D)=\chi_c(D)=\chi(D)=3$.

(ii) If $b-a=3k+1$, then $\chi_f(D)=\chi_c(D)=3+1/(a+k)$, $\chi(D)=4$.

(iii) If $b-a=3k+2$, then $\chi_f(D)=\chi_c(D)=3+1/(a+2k+1)$, $\chi(D)=4$.

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