

# Nonlinear Boundary Value Problems for a Class of Quasilinear Integrodifferential Equations

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**Abstract** In this paper , the existence of solution for a class of quasilinear integrodifferential equations with nonlinear boundary conditions

$$\begin{aligned} &(\Phi_p(u'))' = f(t, u, T_1 u, T_2 u, u') \\ &I(u(0), u(1)) = 0, \quad R(u(0), u(1), u'(0), u'(1)) = 0 \end{aligned}$$

is established , which occurs in the study of the  $p$ -Laplace equation , generalized reaction-diffusion theory , non-Newtonian fluid theory , and the turbulent flow of a gas in porous medium. The results are obtained by using upper and lower solutions method. The results of this paper are new and extend previously known results.

**Key words** quasilinear integrodifferential equation , nonlinear boundary value problem , upper and lower solutions  
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## 一类拟线性微分积分方程非线性边值问题

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[ 摘要 ] 本文研究了下面一类拟线性积分微分方程非线性边值问题

$$\begin{aligned} &(\Phi_p(u'))' = f(t, u, T_1 u, T_2 u, u') \\ &I(u(0), u(1)) = 0, \quad R(u(0), u(1), u'(0), u'(1)) = 0 \end{aligned}$$

解的存在性 , 此类问题来自于研究  $p$ -拉普拉斯方程 , 一般化的反应扩散理论 , 非牛顿流体理论和多孔介质中的气体湍流等问题. 所得结果是利用上下解方法得到. 本文的结果是新的且推广了已知结果.

[ 关键词 ] 拟线性积分微分方程 , 非线性边值问题 , 上下解

## 0 Introduction

We consider the nonlinear equation

$$(\Phi_p(u'))' = f(t, u, T_1 u, T_2 u, u'), \quad t \in I = [0, 1] \tag{1}$$

satisfying the conditions

$$\begin{cases} I(u(0), u(1)) = 0, \\ R(u(0), u(1), u'(0), u'(1)) = 0, \end{cases} \tag{2}$$

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where

$$T_1 u(t) = \psi_1(t) + \int_0^t K_1(t,s)u(s)ds, \quad T_2 u(t) = \psi_2(t) + \int_0^t K_2(t,s)u(s)ds,$$

$K_i(t,s) \in C([0,1] \times [0,1], \mathbf{R}^+)$ ,  $\psi_i(t) \in C([0,1], \mathbf{R})$ ,  $i=1,2$ ,  $f: [0,1] \times \mathbf{R}^4 \rightarrow \mathbf{R}$ ,  $L: \mathbf{R}^2 \rightarrow \mathbf{R}$  and  $R: \mathbf{R}^4 \rightarrow \mathbf{R}$  are continuous functions,  $\Phi_p(s) = |s|^{p-2}s$ ,  $p > 1$ . Equations of the above form are mathematical models occurring in studies of the  $p$ -Laplace equation, generalized reaction-diffusion theory ([1]), non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium ([2]). In the non-Newtonian fluid theory, the quantity  $p$  is characteristic of the medium. Media with  $p > 2$  are called dilatant fluids and those with  $p < 2$  are called pseudoplastics. If  $p = 2$ , they are Newtonian fluids.

For the equation

$$(\Phi_p(u'))' = f(t, u, u'), \quad t \in I = [0,1] \quad (3)$$

with different boundary conditions has been studied by many authors (see [3–13] and references therein). On the contrary, it seems that little is known about the result for problem (1)–(2). Recently, Yang [14] studied existence results for the integrodifferential equation (1) subject to one of the following boundary conditions:

$$g(u(0))\mu(1)\mu'(0)\mu'(1) = 0, \quad h(u(0)) = u(1)$$

or

$$p(u(0)\mu'(0)) = 0 = q(u(0)\mu'(0)\mu(1)\mu'(1))$$

or

$$r(u(1)\mu'(1)) = 0 = u(u(0)\mu'(0)\mu(1)\mu'(1)).$$

However, the periodic case is not included in above boundary conditions. We give the existence of solutions for the boundary value problems (1)–(2), which contains the boundary value problem with periodic boundary condition  $u(0) = u(1)$ ,  $u'(0) = u'(1)$ . Our results of this paper extend and complement results of the [3–4, 14–17].

**Definition 1** We say that a function  $\alpha(t) \in C^1[0,1]$  is a lower solution of equation (3) if  $\Phi_p(\alpha') \in C^1(0,1)$  and satisfies

$$(\Phi_p(\alpha'))' \geq f(t, \alpha, \alpha'), \quad \text{for a. e. } t \in I.$$

Analogously, we say that  $\beta \in C^1[0,1]$  is an upper solution of equation (3) if  $\Phi_p(\beta') \in C^1[0,1]$  and satisfies

$$(\Phi_p(\beta'))' \leq f(t, \beta, \beta'), \quad \text{for a. e. } t \in I.$$

In what follows we shall assume that

$$\alpha(t) \leq \beta(t), \quad t \in I.$$

We define the set

$$E = \{u \in C^1(I) \mid \alpha(t) \leq u(t) \leq \beta(t), \forall t \in I\}$$

Suppose that  $f(t, y, z)$  satisfies the following conditions:

$(H_1)$   $f(t, y, z)$  is a continuous function in  $\Omega = \{(t, y, z) \mid 0 \leq t \leq 1, (y, z) \in \mathbf{R}^2\}$ .

$(H_2)$   $f(t, y, z)$  satisfies Nagumo condition in  $E$ , there exist function  $\Psi: [0, \infty) \rightarrow [0, \infty)$ , and  $\frac{1}{\Psi}$  integrable on each bounded interval  $(a, b) \subset [0, \infty)$ , such that

$$|f(t, y, z)| \leq \Psi(|z|) \quad \text{for } (t, y) \in E, z \in \mathbf{R},$$

where  $\Psi$  satisfies

$$\int_0^\infty \Phi_p^{-1}(u) \Psi(\Phi_p^{-1}(u)) du = \infty.$$

From [8, 10], we have the following theorem

**Theorem 1** Let  $\alpha(t)$  and  $\beta(t)$  be a lower and an upper solution, respectively, for equation (3) and such that  $\alpha \leq \beta$  in  $I$ . Assume that hypotheses  $(H_1)$ – $(H_2)$  are satisfied. Then equation (3) and  $u(0) = A$ ,  $u(1) = B$  has at least one solution  $\alpha(t) \leq u(t) \leq \beta(t)$  for all  $\alpha(0) \leq A \leq \beta(0)$ ,  $\alpha(1) \leq B \leq \beta(1)$ .

# 1 Main Results

Before introducing the main result of existence solutions of problem (1)–(2), firstly, we give the following definition

**Definition 2** We say that a function  $\alpha(t) \in C^1[0, 1]$  is a lower solution of equation (1) if  $\Phi_p(\alpha') \in C^1(0, 1)$  and satisfies

$$(\Phi_p(\alpha'))' \geq f(t, \alpha, T_1\alpha, T_2\alpha, \alpha'), \text{ for a. e. } t \in I.$$

Analogously, we say that  $\beta(t) \in C^1[0, 1]$  is an upper solution of equation (1) if  $\Phi_p(\beta') \in C^1[0, 1]$  and satisfies

$$(\Phi_p(\beta'))' \leq f(t, \beta, T_1\beta, T_2\beta, \beta'), \text{ for a. e. } t \in I.$$

Second, assume that  $f(t, \mu, \nu, w, z)$  satisfies the following conditions:

(H<sub>3</sub>)  $f(t, \mu, \nu, w, z)$  is nonincreasing in  $v$  and in  $w$  respectively.

(H<sub>4</sub>)  $f(t, \mu, \nu, w, z) \in C([0, 1] \times \mathbf{R}^4, \mathbf{R})$ , for any positive constants  $r_1, r_2, r_3 > 0$ , there exists a positive function  $h(x) \in C[0, \infty)$  satisfying

$$\int_0^\infty \Phi_p^{-1}(u)/h(\Phi_p^{-1}(u))du = \infty.$$

and while  $0 \leq t \leq 1, |u| \leq r_1, |v| \leq r_2, |w| \leq r_3, z \in \mathbf{R}, |f(t, \mu, \nu, w, z)| \leq h(|z|)$ .

Now, we can prove our main results

**Theorem 2** Let  $\alpha(t)$  and  $\beta(t)$  be a lower and an upper solution, respectively, for equation (1) and such that  $\alpha(t) \leq \beta(t)$  in  $I$ . Assume that hypotheses (H<sub>3</sub>)–(H<sub>4</sub>) are satisfied. Then boundary value problem

$$(\Phi_p(u'))' = f(t, \mu, T_1u, T_2u, u'), u(0) = A, u(1) = B \quad (4)$$

has at least one solution  $\alpha(t) \leq u(t) \leq \beta(t)$  for all  $\alpha(0) \leq A \leq \beta(0), \alpha(1) \leq B \leq \beta(1)$ .

**Proof** Let  $u_0(t) = \beta(t)$ , then

$$\begin{aligned} f(t, \alpha(t), [T_1u_0](t), [T_2u_0](t), \alpha'(t)) &\leq f(t, \alpha(t), [T_1\alpha](t), [T_2\alpha](t), \alpha'(t)) \leq (\Phi_p(\alpha'))' \\ f(t, \beta(t), [T_1u_0](t), [T_2u_0](t), \beta'(t)) &\geq (\Phi_p(\beta'))', t \in I = [0, 1]. \end{aligned}$$

From Theorem 1, we have that the problem

$$(\Phi_p(u'))' = f(t, \mu, T_1u_0, T_2u_0, u'), u(0) = A, u(1) = B$$

has at least one solution  $u_1(t)$  such that  $\alpha(t) \leq u_1(t) \leq \beta(t) = u_0(t)$ .

We consider the problem again

$$(\Phi_p(u'))' = f(t, \mu, T_1u_1, T_2u_1, u'), u(0) = A, u(1) = B.$$

Clearly,

$$f(t, \alpha, T_1u_1, T_2u_1, \alpha') \leq (\Phi_p(\alpha'))',$$

and

$$f(t, \mu_1, T_1u_1, T_2u_1, \mu_1') \geq f(t, \mu_1, T_1u_0, T_2u_0, \mu_1') = (\Phi_p(u_1'))'.$$

From Theorem 1, the above problem has at least one solution  $u_2(t)$  such that  $\alpha(t) \leq u_2(t) \leq u_1(t)$ .

By the induction principle, we can construct a nonincreasing sequence  $\{u_n(t)\}$  such that

$$\alpha(t) \leq u_n(t) \leq u_{n-1}(t) \leq \dots \leq u_0(t) = \beta(t).$$

First let  $M > 0$  such that  $|\alpha(t)| \leq M, |\beta(t)| \leq M, t \in [0, 1] = I$ , then  $|u_n(t)| \leq M$ . Next we shall prove  $|u'_n(t)| \leq N$ . From condition (H<sub>4</sub>), we obtain

$$\int_{(2M)^{p-1}}^{N^{p-1}} \frac{\Phi_p^{-1}(u)}{h(\Phi_p^{-1}(u))} du > 2M \quad (5)$$

where  $N > \max\{2M, \max_{t \in I}\{|\beta'(t)|, |\alpha'(t)|\}\}$ . Suppose not, there is  $t_2 \in [0, 1]$  such that  $|u'(t_2)| > N$ . We can suppose  $u'(t_2) > N$ . From the Mean Value Theorem, there is an  $\zeta \in [0, 1]$  such that  $u(1) - u(0) = u'(\zeta)$ . Thus  $|u'(\zeta)| \leq 2M < N$ . Then there is interval  $[c, d] \subset [0, 1]$  such that  $u'(c) = 2M, u'(d) = N$ , and if  $t \in [c, d], 2M \leq u' \leq N$ . From (4) which implies that  $(\Phi_p(u'))' \leq h(u')$ . Then we have

$$\int_c^d \frac{\Phi_p^{-1}(\Phi_p(u')) \chi(\Phi_p(u'))'}{h(\Phi_p^{-1}(\Phi_p(u')))} ds \leq \int_c^d \frac{u'(\Phi_p(u'))'}{h(u')} ds \leq \int_c^d u' ds \leq 2M.$$

Therefore , we have

$$\int_{(2M)^{p-1}}^{N^{p-1}} \frac{\Phi_p^{-1}(u)}{h(\Phi_p^{-1}(u))} du \leq 2M.$$

This is a contradiction in view of (5). Thus we have  $|u'(t)| \leq N$ . On the other hand,  $u_n(t)$  satisfies equation , we have  $\{(\Phi_p(u'_n))'\}$  which is uniformly bounded on  $I$ . Therefore,  $\{u_n(t)\}, \{\Phi_p(u'_n)\}$  are uniformly bounded and equicontinuous. By Arzela-Ascoli Theorem, there exists a subsequence  $\{u_{n_k}\}$  satisfying  $\lim_{k \rightarrow \infty} \Phi_p(u'_{n_k}) = v$ . Thus, we obtain

$$\lim_{k \rightarrow \infty} u'_{n_k} = \Phi_p^{-1}(v) \Rightarrow u_{n_k} = A + \int_0^t u'_{n_k}(s) ds \rightarrow A + \int_0^t \Phi_p^{-1}(v) ds = \bar{u} \quad (k \rightarrow \infty).$$

Then there exists  $\bar{u} \in C^1(I)$  such that  $\lim_{k \rightarrow \infty} u_{n_k}(t) = \bar{u}$ . From the dominated convergence theorem, we conclude that  $\bar{u}$  is a solution of problem (4).

**Theorem 3** Assume that

(1)  $(H_3)-(H_4)$  are satisfied;

(2)  $(\zeta, \eta) \in \mathcal{Q}(\mathbf{R}^2)$  and there exist nondecreasing functions  $\chi(s)$  and  $\theta(s)$ , which are continuous on  $\mathbf{R}$ , such that  $(\chi(s), \theta(s)) \equiv 0$  for  $s \in \mathbf{R}$ ;

(3)  $(\zeta, \eta, \mu, \nu) \in \mathcal{Q}(\mathbf{R}^4)$ , and nondecreasing with respect to  $u$ , nonincreasing with respect to  $v$ ;

(4) let  $\alpha(t)$  and  $\beta(t)$  be a lower and an upper solution, respectively, for equation (1) and satisfy the following hypotheses

$$R(\beta(0), \beta(1), \beta'(0), \beta'(1)) \leq 0 \leq R(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1)),$$

and there exist  $s_1$  and  $s_2$  ( $s_1 \leq s_2$ ) such that

$$\chi(s_1) = \alpha(0), \theta(s_1) = \alpha(1), \chi(s_2) = \beta(0), \theta(s_2) = \beta(1).$$

Then the boundary value problem (1)-(2) has a solution  $u(t)$  such that  $\alpha(t) \leq u(t) \leq \beta(t)$  in  $I$ .

**Proof** By the monotonicity of  $\chi(s)$  and  $\theta(s)$ , for arbitrary  $s_1 \leq s \leq s_2$  we have  $\alpha(0) \leq \chi(s) \leq \beta(0)$ ,  $\alpha(1) \leq \theta(s) \leq \beta(1)$ . Hence, by Theorem 2, the problem

$$(\Phi_p(u'))' = f(t, \mu, T_1 u, T_2 u, u'), u(0, s) = \chi(s), u(1, s) = \theta(s)$$

has at least one solution  $\alpha(t) \leq u(t, s) \leq \beta(t)$ .

If  $s = s_1$ , then  $u(0, s_1) = \alpha(0)$ ,  $u(1, s_1) = \alpha(1)$ . Since  $u'(0, s_1) \geq \alpha'(0)$ ,  $u'(1, s_1) \leq \alpha'(1)$ . Hence

$$\begin{aligned} R(u(0, s_1), u(1, s_1), u'(0, s_1), u'(1, s_1)) &= R(\alpha(0), \alpha(1), u'(0, s_1), u'(1, s_1)) \\ &\geq R(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1)) \geq 0. \end{aligned} \quad (6)$$

Similarly if  $s = s_2$ , then we have

$$u(0, s_2) = \beta(0), u(1, s_2) = \beta(1); u'(0, s_2) \leq \beta'(0), u'(1, s_2) \geq \beta'(1).$$

hence

$$\begin{aligned} R(u(0, s_2), u(1, s_2), u'(0, s_2), u'(1, s_2)) &= R(\beta(0), \beta(1), u'(0, s_2), u'(1, s_2)) \\ &\leq R(\beta(0), \beta(1), \beta'(0), \beta'(1)) \leq 0. \end{aligned} \quad (7)$$

Define the sets  $\Omega_1, \Omega_2$  by

$$\Omega_1 = \{s : R(u(0, s), u(1, s), u'(0, s), u'(1, s)) > 0, s_1 \leq s \leq s_2\},$$

$$\Omega_2 = \{s : R(u(0, s), u(1, s), u'(0, s), u'(1, s)) < 0, s_1 \leq s \leq s_2\}.$$

From (6) and (7), we know that  $\Omega_1 \cap \Omega_2 = \emptyset$ . We may show that  $\Omega_1$  and  $\Omega_2$  are closed. Thus there exists an  $s_0 \in [s_1, s_2]$  such that  $R(u(0, s_0), u(1, s_0), u'(0, s_0), u'(1, s_0)) = 0$ . The proof of Theorem 3 is complete.

**Remark 1** If the boundary condition (2) is written in the following form

$$I(u(0), u'(0)) = 0, R(u(0), u'(0), u(1), u'(1)) = 0 \quad (8)$$

then we have a similar results with Theorem 3.

**Example** To illustrate Theorem 3 for the case when the boundary conditions are nonlinear, let satisfy con-

ditions  $(H_3) - (H_4)$  and assume

$$\begin{aligned} & f(t, -1) \mu_1(t) + \int_0^t K_1(t, s) ds \mu_2(t) + \int_0^t K_2(t, s) ds \rho \leq 0 \\ & \leq f(t, 1) \mu_1(t) + \int_0^t K_1(t, s) ds \mu_2(t) + \int_0^t K_2(t, s) ds \rho \quad \rho \leq t \leq 1 \end{aligned}$$

where  $K_i \in C([0, 1] \times [0, 1], \mathbb{R}^+)$ ,  $\mu_i \in C([0, 1], \mathbb{R})$ ,  $i = 1, 2$ , so that  $\alpha = -1$ ,  $\beta = 1$  are lower and upper solutions, respectively, of (1). Let  $L = L(\zeta, \eta)$ ,  $R = R(\zeta, \eta, \mu, \nu)$  be defined by

$$L(\zeta, \eta) = \zeta - \eta, \quad R(\zeta, \eta, \mu, \nu) = u - v.$$

By Theorem 3, there exists a solution  $u$  of the boundary value problem

$$\begin{aligned} & (\Phi_p(u'))' = f(t, \mu, T_1 u, T_2 u, u'), \\ & u(0) = u(1), \quad u'(0) = u'(1), \end{aligned}$$

satisfying  $-1 \leq u(x) \leq 1$  on  $[0, 1]$ .

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