# Meromorphic Functions That Share Rational Functions

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**Abstract** In this paper, the uniqueness of meromorphic functions is studied and the following result is proved. Let $p(z)$ and $q(z)$ be two coprime polynomials of degree $n_1$ and $n_2$ respectively, let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, and let $n \geqslant \max\{11, 2n_1 + 4n_2 + 3\}$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $p(z)/q(z)$ CM, then $f(z) = c_1 Q(z) e^{\alpha(z)}$, $g(z) = c_2 Q^{-1}(z) e^{-\alpha(z)}$, where $c_1$, $c_2$ are two constants, $Q(z)$ is a rational function, and $\alpha(z)$ is a nonconstant polynomial satisfying

$$(c_1c_2)^{n+1}\left(\frac{Q'(z)}{Q(z)} + \alpha'(z)\right)^2 \equiv -\left(\frac{p(z)}{q(z)}\right)^2,$$ or $f(z) \equiv tg(z)$ for a constant $t$ satisfying $t^{n+1} = 1$.

**Key words** meromorphic function, entire function, rational function, constant, uniqueness

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# 分担有理函数的亚纯函数

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[摘要] 研究亚纯函数的惟一性, 证明如下结果: 设 $p(z)$ 和 $q(z)$ 分别为 $n_1$ 和 $n_2$ 次多项式且互素, $f(z)$ 和 $g(z)$ 是两个超越亚纯函数, $n \geqslant \max\{11, 2n_1 + 4n_2 + 3\}$ 是一个正整数, 如果 $f^n(z)f'(z)$, $g^n(z)g'(z)$ 分担有理函数 $\frac{p(z)}{q(z)}$ CM, 则 $f(z) = c_1Q(z)e^{\alpha(z)}$, $g(z) = c_2Q^{-1}(z)e^{-\alpha(z)}$, 这里 $c_1$, $c_2$ 是两个常数, $Q(z)$ 是一个有理函数, $\alpha(z)$ 是一个非常数多项式, 满足 $(c_1c_2)^{n+1}\left(\frac{Q'(z)}{Q(z)} + \alpha'(z)\right)^2 \equiv -\left(\frac{p(z)}{q(z)}\right)^2$; 或者 $f(z) \equiv tg(z)$, 其中 $t$ 是满足 $t^{n+1} = 1$ 的常数.

[关键词] 亚纯函数, 整函数, 有理函数, 常数, 惟一性

## 0 Introduction

In this paper, by a meromorphic function we always mean a function which is meromorphic in the whole complex plane. Let $f(z)$ be a nonconstant meromorphic function. We use the following standard notations of value distribution theory, $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $N(r, 1/f)$, $\cdots$ (see Hayman[1], Yang[2], Fang[3]). We denote by $S(r, f)$ any function satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to +\infty$, possibly outside of a set $E$ with finite measure.

Let $a$ be a finite complex numbers. We denote by $N_{2)}\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f(z) - a$

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with multiplicity at most 2, and by $N_{2)}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(2}\left(r, \frac{1}{f-a}\right)$ be the counting function for zeros of $f(z)-a$ with multiplicity at least 2 and $\overline{N}_{(2}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Set

$$N_2\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right).$$

Let $f(z)$ and $g(z)$ be two meromorphic functions, and let $p(z)$ and $q(z)$ be two polynomials. If $f(z)-\frac{p(z)}{q(z)}$ and $g(z)-\frac{p(z)}{q(z)}$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share $\frac{p(z)}{q(z)}$ CM. In addition, the sign "$p(z)\mid q(z)$" means that there exists a polynomial $r(z)$ such that $q(z)=r(z)p(z)$. The following results were proved in [4] and [5].

**Theorem A** Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $n\geqslant 11$ a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$, or $f(z)\equiv tg(z)$ for a constant $t$ such that $t^{n+1}=1$.

**Theorem B** Let $f(z)$ and $g(z)$ be two nonconstant meromorphic (entire) functions, $n\geqslant 11$ $(n\geqslant 6)$ a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $z$ CM, then either $f(z)=c_1e^{cz^2}$, $g(z)=c_2e^{-cz^2}$, where $c_1$, $c_2$ and $c$ are three constants satisfying $4(c_1c_2)^{n+1}c^2=-1$, or $f(z)\equiv tg(z)$ for a constant $t$ such that $t^{n+1}=1$.

In this paper, we extend above results as follows.

**Proposition 1** Let $p(z)$, $q(z)$ be two coprime polynomials of degree $n_1$ and $n_2$ respectively; let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, and $n\geqslant\max\{11, 2n_1+4n_2+3\}$ a positive integer. If $f^n(z)f'(z)g^n(z)g'(z)\equiv\left(\frac{p(z)}{q(z)}\right)^2$, then $f(z)=c_1Q(z)e^{\alpha(z)}$, $g(z)=c_2Q^{-1}(z)e^{-\alpha(z)}$, where $c_1$, $c_2$ are two constants, and $Q(z)$ is a rational function, and $\alpha(z)$ is a nonconstant polynomial satisfying

$$(c_1c_2)^{n+1}\left(\frac{Q'(z)}{Q(z)}+\alpha'(z)\right)^2\equiv\left(\frac{p(z)}{q(z)}\right)^2.$$

**Theorem 1** Let $p(z)$ and $q(z)$ be two coprime polynomials of degree $n_1$ and $n_2$ respectively, and $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, $n\geqslant\max\{11, 2n_1+4n_2+3\}$ a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $p(z)/q(z)$ CM, then either $f(z)=c_1Q(z)e^{\alpha(z)}$, $g(z)=c_2Q^{-1}(z)\cdot e^{-\alpha(z)}$, where $c_1$, $c_2$ are two constants, and $Q(z)$ is a rational function, and $\alpha(z)$ is a polynomial satisfying $(c_1c_2)^{n+1}\left(\frac{Q'(z)}{Q(z)}+\alpha'(z)\right)^2\equiv\left(\frac{p(z)}{q(z)}\right)^2$, or $f(z)\equiv tg(z)$ for a constant $t$ such that $t^{n+1}=1$.

## 1 Some Lemmas

In order to prove Theorem 1, we need the following lemmas.

**Lemma 1**[6] Let $a_1$, $a_2$, $\cdots$, $a_n$ be finite complex numbers, $a_n\neq 0$, and let $f$ be a nonconstant meromorphic function. Then $T(r, a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f)=nT(r, f)+S(r, f)$.

**Lemma 2**[7] Let $f_j(z)$ and $g_j(z)$ $(j=1, 2, \cdots, n)$ be two sets of entire functions satisfying:

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$;

(ii) $f_j(z)$ has a smaller order than $e^{g_l(z)-g_k(z)}$, where $1\leqslant j\leqslant n$, $1\leqslant l, k\leqslant n$, and $l\neq k$, then $f_j(z)\equiv 0$ $(j=1, 2, \cdots, n)$.

**Lemma 3**[4,8,9] Let $f(z)$ and $g(z)$ be two meromorphic functions. If $f$ and $g$ share 1 CM, then one of the following cases must occur:

(i) $T(r, f)+T(r, g)\leqslant 2\{N_2(r, \frac{1}{f})+N_2(r, \frac{1}{g})+N_2(r, f)+N_2(r, g)\}+S(r, f)+S(r, g)$;

(ii) either $f\equiv g$ or $fg\equiv 1$.

**Lemma 4** Let $p(z)$ and $q(z)$ be two coprime polynomials of degree $n_1$ and $n_2$. Suppose that $f(z)$, $g(z)$ are two nonconstant meromorphic functions, and $n\geqslant\max\{4, 2n_1+4n_2+3\}$ is a positive integer. If

$$f^n(z)f'(z)g^n(z)g'(z)\equiv\left(\frac{p(z)}{q(z)}\right)^2 \tag{1}$$

and $f(z)g(z)\neq 0$, then $h(z)=\dfrac{1}{f(z)g(z)}$ is a polynomial of degree at most 2.

**proof** By $h(z)=\dfrac{1}{fg}$ and (1) we have

$$\left(\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}\right)^2=\frac{1}{4}\left(\frac{h'}{h}\right)^2-\left(\frac{p}{q}\right)^2h^{n+1}. \tag{2}$$

Let

$$\alpha=\frac{g'}{g}+\frac{1}{2}\frac{h'}{h}. \tag{3}$$

By (2)

$$\alpha^2=\frac{1}{4}\left(\frac{h'}{h}\right)^2-\left(\frac{p}{q}\right)^2h^{n+1}. \tag{4}$$

If $\alpha\equiv 0$, then by (4) we get that

$$h^{n+1}=\frac{1}{4}\left(\frac{q}{p}\right)^2\left(\frac{h'}{h}\right)^2. \tag{5}$$

Since $h$ is an entire, we have

$$(n+1)T(r,h)=(n+1)m(r,h)\leqslant 2m\left(r,\frac{q}{p}\right)+2m\left(r,\frac{h'}{h}\right)\leqslant 2\max\{n_2-n_1,0\}\log r+S(r,h).$$

Thus by $n\geqslant 2n_1+4n_2+3$, we know that $h$ is a constant.

Now we assume that $\alpha\neq 0$. Differentiating two sides of (4), we have

$$2\alpha\alpha'=\frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)'-\left(2\left(\frac{p}{q}\right)\left(\frac{p}{q}\right)'h^{n+1}+(n+1)\left(\frac{p}{q}\right)^2h^nh'\right). \tag{6}$$

It follows from (4) and (6) that

$$h^{n+1}\left(2\left(\frac{p}{q}\right)\left(\frac{p}{q}\right)'+(n+1)\left(\frac{p}{q}\right)^2\frac{h'}{h}-2\left(\frac{p}{q}\right)^2\frac{\alpha'}{\alpha}\right)=\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)'-\frac{h'}{h}\frac{\alpha'}{\alpha}\right). \tag{7}$$

If

$$2\left(\frac{p}{q}\right)\left(\frac{p}{q}\right)'+(n+1)\left(\frac{p}{q}\right)^2\frac{h'}{h}-2\left(\frac{p}{q}\right)^2\frac{\alpha'}{\alpha}\equiv 0,$$

then by (7) we deduce that either $h'/h\equiv 0$ or $(h'/h)'-(h'\alpha')/(h\alpha)\equiv 0$. If $h'/h\equiv 0$, then $h$ is a constant. If $(h'/h)'-(h'\alpha')/(h\alpha)\equiv 0$, then we have

$$h'/h=c\alpha, \tag{8}$$

where $c$ is a constant. If $c=0$ then $h$ is a constant. If $c\neq 0$, then we get from (4) and (8) that

$$h^{n+1}=\left(\frac{q}{p}\right)^2\left(\frac{1}{4}-\frac{1}{c^2}\right)\left(\frac{h'}{h}\right)^2.$$

Since $h$ is an entire, we have

$$(n+1)T(r,h)\leqslant 2m(r,q/p)+2m(r,h'/h)+O(1)\leqslant 2\max\{n_2-n_1,0\}\log r+S(r,h).$$

Then by $n\geqslant 2n_2$ we know that $h$ is a constant.

Next we assume that

$$2\left(\frac{p}{q}\right)\left(\frac{p}{q}\right)'+(n+1)\left(\frac{p}{q}\right)^2\frac{h'}{h}-2\left(\frac{p}{q}\right)^2\frac{\alpha'}{\alpha}\not\equiv 0.$$

Then by (7), the first fundamental theorem, and the fact that $h$ is an entire, we obtain

$$(n+1)T(r,h)=m(r,h^{n+1})\leqslant m\left(r,\frac{1}{2(p/q)(p/q)'+(n+1)(p/q)^2(h'/h)-2(p/q)^2\alpha'/\alpha}\right)$$

$$+ m\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha'}{\alpha}\right)\right) + O(1). \quad (9)$$

By (4) the poles of $\alpha$ must be the zero points of $h$ or $q$. If $n_1 \geqslant n_2$, then we deduce from (9) that

$$\begin{aligned}(n+1)T(r,h) &\leqslant 2m\left(r, \frac{q}{p}\right) + m\left(r, \frac{1}{2(p/q)'/(p/q) + (n+1)(h'/h) - 2\alpha'/\alpha}\right)\\ &+ m\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha'}{\alpha}\right)\right) + O(1)\\ &\leqslant T\left(r, 2\frac{(p/q)'}{p/q} + (n+1)\frac{h'}{h} - 2\frac{\alpha'}{\alpha}\right) + m\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha'}{\alpha}\right)\right) + O(1)\\ &\leqslant (n_1 + n_2)\log r + N\left(r, \frac{1}{h}\right) + N\left(r, \frac{1}{\alpha}\right) + S(r,h) + S(r,\alpha). \end{aligned} \quad (10)$$

If $n_1 < n_2$, then we deduce from (9) that

$$\begin{aligned}(n+1)T(r,h) &\leqslant 2m\left(r, \frac{q}{p}\right) + m\left(r, \frac{1}{2(p/q)'/(p/q) + (n+1)(h'/h) - 2\alpha'/\alpha}\right)\\ &+ m\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha'}{\alpha}\right)\right) + O(1)\\ &\leqslant 2(n_2 - n_1)\log r + T\left(r, 2\frac{(p/q)'}{p/q} + (n+1)\frac{h'}{h} - 2\frac{\alpha'}{\alpha}\right)\\ &+ m\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha'}{\alpha}\right)\right) + O(1)\\ &\leqslant (3n_2 - n_1)\log r + N\left(r, \frac{1}{h}\right) + N\left(r, \frac{1}{\alpha}\right) + S(r,h) + S(r,\alpha). \end{aligned} \quad (11)$$

By Lemma 1 and (4) we have

$$2T(r,\alpha) = T(r,\alpha^2) + S(r,\alpha) = T\left(r, \frac{1}{4}\left(\frac{h'}{h}\right)^2 - \left(\frac{p}{q}\right)^2 h^{n+1}\right) + S(r,\alpha). \quad (12)$$

If $n_1 \geqslant n_2$, then we deduce from (12) that

$$2T(r,\alpha) \leqslant 2n_1\log r + (n+1)T(r,h) + 2N\left(r, \frac{1}{h}\right) + S(r,h) + S(r,\alpha). \quad (13)$$

If $n_1 < n_2$, then we deduce from (12) that

$$2T(r,\alpha) \leqslant 2n_2\log r + (n+1)T(r,h) + 2N\left(r, \frac{1}{h}\right) + S(r,h) + S(r,\alpha). \quad (14)$$

Thus by (10) ~ (14) we obtain

$$\begin{aligned}(n-3)T(r,h) &\leqslant 2(2n_1 + n_2)\log r + S(r,h), \quad (\text{if } n_1 \geqslant n_2),\\ (n-3)T(r,h) &\leqslant 8n_2\log r + S(r,h), \quad (\text{if } n_1 < n_2). \end{aligned} \quad (15)$$

Then by (15), Lemma 4 is proved.

**Lemma 5** Let $p(z)$ and $q(z)$ be two coprime polynomials of degree $n_1$ and $n_2$, $f(z)$ and $g(z)$ be two transcendental meromorphic (entire) functions, and $n \geqslant 11$ $(n \geqslant 6)$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $\frac{p(z)}{q(z)}$ CM, then either $f^n(z)f'(z)g^n(z)g'(z) \equiv \left(\frac{p(z)}{q(z)}\right)^2$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.

**Proof** Let $F(z) = \frac{f^n(z)f'(z)}{p(z)/q(z)}$, $G(z) = \frac{g^n(z)g'(z)}{p(z)/q(z)}$. Then by $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $\frac{p(z)}{q(z)}$ CM, we know that $F(z)$ and $G(z)$ share 1 CM. By the first fundamental theorem and $m(r, 1/f) \leqslant m(r, 1/f') + S(r, f)$ we have

$$N\left(r, \frac{1}{f'}\right) \leqslant N\left(r, \frac{1}{f}\right) + N(r,f) + S(r,f).$$

Hence we obtain

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)&\leqslant 2N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f'}\right)+n_2\log r+S(r,f)\\&=\frac{2}{n}\left(nN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f'}\right)\right)+\left(1-\frac{2}{n}\right)N\left(r,\frac{1}{f'}\right)+n_2\log r+S(r,f)\\&\leqslant\frac{2}{n}N\left(r,\frac{1}{f^nf'}\right)+\left(1-\frac{2}{n}\right)\frac{n}{n+1}\left(N\left(r,\frac{1}{f}\right)+N(r,f)\right)\\&\quad+\left(1-\frac{2}{n}\right)\frac{1}{n+1}N\left(r,\frac{1}{f'}\right)+n_2\log r+S(r,f)\\&\leqslant\frac{2}{n}N\left(r,\frac{1}{f^nf'}\right)+\left(1-\frac{2}{n}\right)\frac{1}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\left(1-\frac{2}{n}\right)\frac{n}{n+1}N(r,f)+n_2\log r+S(r,f)\\&=\frac{3}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\frac{n-2}{n+1}N(r,f)+n_2\log r+S(r,f).\end{aligned}\tag{16}$$

Thus we have

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)+N_2(r,f)&\leqslant\frac{3}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\frac{n-2}{n+1}N(r,f)+2N(r,f)+(n_1+n_2)\log r+S(r,f)\\&\leqslant\frac{3}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\frac{3n}{n+1}N(r,f)+(n_1+n_2)\log r+S(r,f)\\&\leqslant\frac{3}{n+1}N\left(r,\frac{1}{F}\right)+\frac{3n_1}{n+1}\log r+\frac{3n}{(n+1)(n+2)}N(r,f)+\frac{3nn_2}{(n+1)(n+2)}\log r\\&\quad+(n_1+n_2)\log r+S(r,f).\end{aligned}\tag{17}$$

On the other hand, by Lemma 1 we have

$$nT(r,f)=T(r,f^n)+S(r,f)\leqslant T\left(r,\frac{f^nf'}{p/q}\right)+2T(r,f)+(n_1+n_2)\log r+S(r,f),$$

that is

$$(n-2)T(r,f)\leqslant T(r,f)+(n_1+n_2)\log r+S(r,f).\tag{18}$$

Thus by (17) and (18) we have

$$\begin{aligned}N_2\left(r,\frac{1}{F}\right)+N_2(r,f)\leqslant\frac{6}{n+2}T(r,f)+(n_1+n_2)\log r+\frac{3n_1}{n+1}\log r\\+\frac{3nn_2}{(n+1)(n+2)}\log r+S(r,f).\end{aligned}\tag{19}$$

Similarly, we have

$$\begin{aligned}N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\leqslant\frac{6}{n+2}T(r,G)+(n_1+n_2)\log r+\frac{3n_1}{n+1}\log r\\+\frac{3nn_2}{(n+1)(n+2)}\log r+S(r,G).\end{aligned}\tag{20}$$

Suppose that

$$T(r,f)+T(r,G)\leqslant 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2(r,f)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right\}+S(r,f)+S(r,G).\tag{21}$$

Then by (19) ~ (21) and $n\geqslant 11$ we get that

$$T(r,f)+T(r,G)\leqslant\frac{4(n+2)}{n-10}\left(n_1+n_2+\frac{3n_1}{n+1}\right)\log r+\frac{12nn_2}{n+1}\log r+S(r,f)+S(r,G).\tag{22}$$

Since both $f$ and $g$ are transcendental meromorphic function, then both $F$ and $G$ also are transcendental meromorphic function. Therefore by (22) we get a contradiction.

By Lemma 3 we get that either $F(z)G(z)\equiv 1$ or $F(z)\equiv G(z)$, that is either $f^n(z)f'(z)g^n(z)g'(z)\equiv\left(\frac{p(z)}{q(z)}\right)^2$ or $f(z)\equiv tg(z)$ for a constant such that $t^{n+1}=1$. The proof of the Lemma 5 is complete.

## 2 Proofs of Proposition 1 and Theorem 1

**Proof of Proposition 1** By

$$f^n(z)f'(z)g^n(z)g'(z) \equiv \left(\frac{p(z)}{q(z)}\right)^2, \tag{23}$$

we first prove that $f(z)g(z) \neq 0$.

In fact, suppose that $f(z)$ has a zero $z_0$ with multiplicity $m$. Then by (23) and $n \geqslant \max\{11, 2n_1+4n_2+3\}$ we know that $z_0$ is a pole of $g(z)$ (say with multiplicity $l \geqslant 1$). We first prove that $m=l$.

Suppose on the contrary that $m \neq l$. Then

$$(m-l)(n+1) \leqslant 2(n_1+1), \quad (\text{if } m>l), \tag{24}$$

$$(l-m)(n+1) \leqslant 2(n_2-1), \quad (\text{if } m<l), \tag{25}$$

which is impossible, since $n \geqslant 2n_1+4n_2+3$.

Hence we get that $m=l$, that is a $f(z)g(z) \neq 0$.

Set

$$h(z) = \frac{1}{f(z)g(z)}. \tag{26}$$

Then $h(z)$ is an entire function. By Lemma 4 we know that $h(z)=\frac{1}{f(z)g(z)}$ is a polynomial of degree at most 2. Then by (23) we have

$$\frac{f'}{f}\frac{g'}{g} = \left(\frac{p}{q}\right)^2 h^{n+1}. \tag{27}$$

On the other hand, by differentiating $fg=\frac{1}{h}$ we get that

$$\frac{f'}{f} + \frac{g'}{g} = -\frac{h'}{h}. \tag{28}$$

Thus by (27) and (28) we get that

$$\frac{f'(z)}{f(z)} = \frac{p_2(z)}{p_1(z)}, \quad \frac{g'(z)}{g(z)} = \frac{q_2(z)}{q_1(z)}, \tag{29}$$

where both $p_1(z)$, $p_2(z)$ and $q_1(z)$, $q_2(z)$ are two coprime polynomials.

Since $f(z)g(z) \neq 0$ and (29) we can write

$$f(z) = \frac{1}{p_3(z)}Q(z)e^{\alpha(z)}, \quad g(z) = \frac{1}{q_3(z)}Q^{-1}(z)e^{-\alpha(z)}, \tag{30}$$

where $Q(z)$ is a rational function, $\alpha(z)$ is a nonconstant polynomial, $p_3$ and $q_3$ are two coprime polynomials satisfying $\deg p_3 + \deg q_3 \leqslant 2$ and the zeros of $p_3(z)$, $q_3(z)$ are not the zeros or poles of $Q(z)$.

Next we prove that both $p_3(z)$ and $q_3(z)$ are constants. By (23) and (30) we have

$$\frac{\left[\left(\alpha'+\frac{Q'}{Q}\right)p_3 - p_3'\right]\left[\left(\alpha'+\frac{Q'}{Q}\right)q_3 + q_3'\right]}{p_3^{n+2}q_3^{n+2}} \equiv -\left(\frac{p(z)}{q(z)}\right)^2. \tag{31}$$

By (31) we know that both $\left[\left(\alpha'+\frac{Q'}{Q}\right)p_3 - p_3'\right]q(z)$ and $\left[\left(\alpha'+\frac{Q'}{Q}\right)q_3 + q_3'\right]q(z)$ are polynomials.

Set

$$p_4(z) = \left[\left(\alpha'+\frac{Q'}{Q}\right)p_3 - p_3'\right]q(z), \quad q_4(z) = \left[\left(\alpha'+\frac{Q'}{Q}\right)q_3 + q_3'\right]q(z). \tag{32}$$

By (31) we have

$$\frac{p_4(z)q_4(z)}{p_3^{n+2}q_3^{n+2}} \equiv -p^2(z). \tag{33}$$

Suppose that $\deg p_3 = m \neq 0$ or $\deg q_3 = l \neq 0$. Without loss of generality, we assume that $\deg p_3 \geqslant \deg q_3$.

Then by $\deg p_3 + \deg q_3 \leqslant 2$ we consider three cases.

**Case 1**　$\deg p_3 = 2$ and $\deg q_3 = 0$. Then by (32) we deduce $\deg p_4 = \deg q_4 + 2$. By (32), (33) and $p_3^{n+2} | p_4 q_4$ we have $p_3^{n+1} | q_4 q$, so that $\deg(q_4 q) \geqslant 2(n+1)$, $\deg(p_4 q) \geqslant 2(n+1) + 2$. Then by $n \geqslant \max\{11, 2n_1 + 4n_2 + 3\}$ we have $\deg(p_4 q_4) - \deg\left(p_3^{n+2} q_3^{n+2}\right) \geqslant \deg(p_4 q_4 q^2) - \deg q^2 - \deg\left(p_3^{n+2} q_3^{n+2}\right) \geqslant 4(n+1) + 2 - 2n_2 - 2(n+2) = 2n + 2 - 2n_2 > 2n_1 = \deg p^2$, which is impossible.

**Case 2**　$\deg p_3 = 1$ and $\deg q_3 = 1$. By (32), (33) and $p_3^{n+2} | p_4 q_4$ we have $p_3^{n+2} | q_4 q$. Similarly we have $q_3^{n+2} | p_4 q$. So that $p_3^{n+3} \mid p_3 q_4 q$, $q_3^{n+3} \mid q_3 p_4 q$.

By $p_3 q_4 q = \left((\alpha' + \frac{Q'}{Q}) p_3 q_3 + p_3 q_3'\right) q^2$ and $q_3 p_4 q = \left((\alpha' + \frac{Q'}{Q}) p_3 q_3 - p_3' q_3\right) q^2$ we know $(p_3 q_4 q)^{(2+2n_2)} = (q_3 p_4 q)^{(2+2n_2)}$. Since $p_3^{n+3-(2+2n_2)} \mid (p_3 q_4 q)^{(2+2n_2)}$, $q_3^{n+3-(2+2n_2)} \mid (q_3 p_4 q)^{(2+2n_2)}$, and $p_3$, $q_3$ are two coprime polynomials, we get that $(p_3 q_3)^{n+3-(2+2n_2)} \mid (p_3 q_4 q)^{(2+2n_2)}$. Thus $\deg(p_3 q_4 q) = \deg(q_3 p_4 q) \geqslant 2\left((n+3) - 2 - 2n_2\right) + 2 + 2n_2$. By $n \geqslant \max\{11, 2n_1 + 4n_2 + 3\}$ we have $\deg(p_4 q_4) - \deg p_3^{n+2} q_3^{n+2} = \deg(p_4 q_4 p_3 q_3 q^2) - 2n_2 - \deg p_3^{n+3} q_3^{n+3} = 2\deg(p_3 q_4 q) - 2n_2 - 2(n+3) \geqslant 2\{2[(n+3) - 2 - 2n_2] + 2 + 2n_2\} - 2n_2 - 2(n+3) = 2(n+3) - 4 - 6n_2 = 2n + 2 - 6n_2 > 2n_1 = \deg p^2$, which is impossible.

**Case 3**　$\deg p_3 = 1$ and $\deg q_3 = 0$. Using the same argument as above Case 2, we can get a contradiction. Hence we deduce that both $p_3(z)$ and $q_3(z)$ are constants. From (23) and (30) we get that $f(z) = c_1 Q(z) e^{\alpha(z)}$, $g(z) = c_2 Q^{-1}(z) e^{-\alpha(z)}$, where $c_1$ and $c_2$ are two nonzero constants, $Q(z)$ is a rational function, and $\alpha(z)$ is a nonconstant polynomial satisfying $(c_1 c_2)^{n+1}\left(\alpha'(z) + \frac{Q'(z)}{Q(z)}\right)^2 \equiv -\left(\frac{p(z)}{q(z)}\right)^2$. The proof of the Proposition 1 is complete.

**Proof of Theorem 1**　By Lemma 5 we have

$$f^n(z) f'(z) g^n(z) g'(z) \equiv \left(\frac{p(z)}{q(z)}\right)^2 \text{ or } f(z) \equiv tg(z) \text{ for a constant such that } t^{n+1} = 1.$$

If $f^n(z) f'(z) g^n(z) g'(z) \equiv \left(\frac{p(z)}{q(z)}\right)^2$, then by Proposition 1 we get that $f(z) = c_1 Q(z) e^{\alpha(z)}$, $g(z) = c_2 Q^{-1}(z) e^{-\alpha(z)}$, where $c_1$ and $c_2$ are two nonzero constants, $Q(z)$ is a rational function, and $\alpha(z)$ is a nonconstant polynomial satisfying $(c_1 c_2)^{n+1}\left(\alpha'(z) + \frac{Q'(z)}{Q(z)}\right)^2 \equiv -\left(\frac{p(z)}{q(z)}\right)^2$. Thus the proof of the Theorem 1 is complete.

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