Note on Two Common Fixed Point Theorems Under Strict Contractive Conditions

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Abstract In this paper, we point out that two common fixed point theorems of non-compatible mappings under strict contractive conditions given by R. P. Pant and V. Pant are incorrect by a counter example. At the same time, we correct these two theorems, and get two new common fixed point theorems of non-compatible mappings.

Key words common fixed points, strict contractive conditions, noncompatible mappings, pointwise R-weak commutativity

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两个严格压缩条件下的公共不动点定理的注记

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[摘要] 本文通过一个反例指出, R. P. Pant和 V. Pant给出的两个在严格压缩条件下的不相容映射的公共不动点定理是不正确的. 同时, 修正了这两个定理,得到两个新的不相容映射的公共不动点定理.

[关键词] 公共不动点, 严格压缩条件, 不相容映射, 点式 R-弱交换性

0 Introduction

In 1986, Jungck[1] introduced the concept of compatible mappings and proved some common fixed point theorems of compatible mappings. However the study of common fixed points of noncompatible mappings is also very interesting[2,3]. In 2000, Pant[4] gave two new common fixed point theorems of noncompatible mappings under strict contractive conditions by using the notion of R-weak commutativity. The aim of this note is to point out that these theorems are incorrect and correct them.

We recall some basic concepts which will be needed in the sequel.

Two selfmaps f, g of a metric space (X, d) are called R-weakly commuting[5] if there exits some real number $R>0$ such that $d(fgx, gfx)\leqslant Rd(fx, gx)$ for all x in X. f and g are called pointwise R-weakly commuting if give x in X, there exists $R>0$ such that $d(fgx, gfx)\leqslant Rd(fx, gx)$. Pant[2,3] proved that pointwise R-weak commutativity is equivalent to commutativity at coincidence points (i.e. weak compatibility defined by Jungck recently[6]).

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Two selfmaps f and g of (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$. f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X, but $\lim_n d(fgx_n, gfx_n)$ is either non-zero or non-existent. Obviously, compatibility implies pointwise R-weak commutativity. However, pointwise R-weakly commuting maps need not be compatible.

1 A Counterexample

In [4], Pant proved the following two common fixed point theorems of noncompatible mappings.

Theorem A Let f and g be noncompatible and pointwise R-weakly commuting selfmaps of a metric space (X, d) such that

(i) $f(X) \subset g(X)$.

(ii) $d(fx, fy) < \max\{d(gx, gy), k[d(fx, gx) + d(fy, gy)]/2, k[d(fy, gx) + d(fx, gy)]/2\}$, $1 \leqslant k < 2$, $x \neq y$.

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point.

Theorem B Let (A, S) and (B, T) be pointwise R-weakly commuting selfmaps of a metric space (X, d) satisfying the conditions

(1) $A(X) \subset T(X)$, $B(X) \subset S(X)$.

(2) $d(Ax, By) < \max\{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2, k[d(Ax, Ty) + d(By, Sx)]/2\}$, $1 \leqslant k < 2$, $x \neq y$.

Let (A, S) or (B, T) be a noncompatible pair of mappings. If the range of one of the mappings is a complete subspace of X, then A, B, S, and T have a unique common fixed point.

The following example shows that when $1 < k < 2$, Theorem A and Theorem B are not valid.

Counterexample Let $X = [2, 19]$ and d be the usual metric on X. We take $k = 1.6$. Define $f, g: X \to X$ by

$$fx = \begin{cases} 3, & \text{if } x = 2 \text{ or } x > 5 \\ 2, & \text{if } x = 3 \\ 6, & \text{if } 2 < x < 3 \text{ or } 3 < x \leqslant 5 \end{cases}$$

and

$$gx = \begin{cases} 3, & \text{if } x = 2 \\ 2, & \text{if } x = 3 \\ 8, & \text{if } 2 < x < 3 \text{ or } 3 < x \leqslant 5 \\ \dfrac{x+1}{2}, & \text{if } x > 5 \end{cases}$$

respectively. Obviously, f and g have not any common fixed point. But, we can prove that f and g satisfy all the conditions of Theorem A.

(1) f and g are pointwise R-weakly commuting since f and g are commuting at their coincidence points $x = 2, 3$.

(2) f and g are noncompatible. In fact, consider the sequence $\{x_n\}$ in X, $x_n = 5 + \frac{1}{n}$. Then we have $\lim_n fx_n = \lim_n gx_n = 3$ but $\lim_n fgx_n = 6$, and $\lim_n gfx_n = 2$. Hence f and g are noncompatible.

(3) $f(X) = \{2, 3, 6\}$, $g(X) = \{2\} \cup [3, 10]$. Hence $f(X)$, $g(X)$ is a complete subspace of X and $f(X) \subset g(X)$.

(4) Take $k = 1.6$, it is easy to verify that f and g satisfy the condition (ii) of Theorem A.

This shows that Theorem A is incorrect when $1 < k < 2$.

Remark 1 In the above example, we take $A = B = f$ and $S = T = g$, then it shows that Theorem B is also incorrect when $1< k< 2$.

Remark 2 From the proofs of Theorem 2.1 and Theorem 2.3 in [3], it is not difficult to see that the theorems are valid when $k= 1$.

2 Revisions of Theorem A and Theorem B

Now we give the correctional forms of Theorem A and Theorem B.

Theorem 1 Let (A, S) and (B, T) be pointwise $R-$ weakly commuting selfmaps of a metric space (X, d) satisfying the conditions:

(i) $A(X)\subset T(X)$, $B(X)\subset S(X)$.

(ii) $d(Ax, By)< \max\{d(Sx, Ty), k[d(Ax, Sx)+d(By, Sx)]/2, k[d(Ax, Ty)+d(By, Ty)]/2\}$, $1\leqslant k < 2$, $x\neq y$.

Let (A, S) or (B, T) be a noncompatible pair of mappings. If the range of one of the mappings is a complete subspace of X, then A, B, S, and T have a unique common fixed point.

Proof Let B and T be noncompatible mappings. Then there exists a sequence $\{x_n\}$ in X such that $Bx_n\to t$ and $Tx_n\to t$ for some t in X, but $\lim_n d(BTx_n, TBx_n)$ is either non-zero or non-existent. Since $B(X)\subset S(X)$, for each x_n there exists y_n in X such that $Bx_n = Sy_n$. Thus we have $Sy_n\to t$. We claim that $Ay_n\to t$. If not, this implies that $d(Ay_n, Bx_n)\not\to 0$, and so there exist a $\varepsilon_0 > 0$ and a subsequence $\{Ay_{n_m}\}$ of $\{Ay_n\}$ such that $d(Ay_{n_m}, Bx_{n_m})\geqslant\varepsilon_0$ $(m = 1, 2, \ldots\ldots)$. Notice that $\lim_n Tx_n = \lim_n Sy_n = t$, hence there exists a positive integer N such that for each $m\geqslant N$ we have

$$d(Bx_{n_m}, Tx_{n_m}) = d(Sy_{n_m}, Tx_{n_m}) < \frac{(2-k)\varepsilon_0}{2k}(< \varepsilon_0),$$

$$d(Ay_{n_m}, Tx_{n_m}) + d(Bx_{n_m}, Tx_{n_m})\leqslant d(Ay_{n_m}, Sy_{n_m}) + d(Sy_{n_m}, Tx_{n_m}) + d(Bx_{n_m}, Tx_{n_m})$$

$$< d(Ay_{n_m}, Sy_{n_m}) + 2\cdot\frac{(2-k)\varepsilon_0}{2k} < \frac{2}{k}d(Ay_{n_m}, Sy_{n_m}).$$

and so by the condition (ii) we obtain

$$\begin{aligned}d(Ay_{n_m}, Bx_{n_m}) &< \max\{d(Sy_{n_m}, Tx_{n_m}), k[d(Ay_{n_m}, Sy_{n_m}) + d(Bx_{n_m}, Sy_{n_m})]/2,\\ &\qquad k[d(Ay_{n_m}, Tx_{n_m}) + d(Bx_{n_m}, Tx_{n_m})]/2\}\\ &\leqslant\max\{\varepsilon_0, d(Ay_{n_m}, Sy_{n_m}), d(Ay_{n_m}, Sy_{n_m})\} = d(Ay_{n_m}, Bx_{n_m}),\end{aligned}$$

a contradiction. Hence $Ay_n\to t$.

Suppose that $S(X)$ is a complete subspace of X. Then, since $Sy_n\to t$, there exists a point u in X such that $t= Su$. If $Au\neq Su$, by (ii) we have

$$d(Au, Bx_n) < \max\{d(Su, Tx_n), k[d(Au, Su) + d(Bx_n, Su)]/2, k[d(Au, Tx_n) + d(Bx_n, Tx_n)]/2\}.$$

Letting $n\to\infty$, it follows that $d(Au, Su) < k[d(Au, Su)]/2< d(Au, Su)$, a contradiction. Hence $Au = Su$. Since (A, S) is pointwise $R-$ weakly commuting, A and S are commuting at coincidence point u, and so $AAu = ASu = SAu = SSu$. Since $A(X)\subset T(X)$, there exists a point w in X such that $Au= Tw$. We assert that $Tw = Bw$. If $Tw\neq Bw$, then by (ii) we get

$$\begin{aligned}d(Au, Bw) &< \max\{d(Su, Tw), k[d(Au, Su) + d(Bw, Su)]/2, d(Au, Tw) + d(Bw, Tw)]/2\}\\ &= k[d(Bw, Au)]/2< d(Bw, Au), \quad \text{as } 1\leqslant k< 2,\end{aligned}$$

a contradiction. Hence $Bw = Tw = Au = Su$. Pointwise $R-$ weak commutativity of (B, T) implies that $BTw = TBw = TTw = BBw$. Now if $Au\neq AAu$, then by (ii) we get

$$\begin{aligned}d(Au, AAu) &= d(AAu, Bw)\\ &< \max\{d(SAu, Tw), k[d(AAu, SAu) + d(Bw, SAu)]/2, k[d(AAu, Tw) + d(Bw, Tw)]/2\}\\ &= \max\{d(AAu, Au), k[0+ d(Au, AAu)]/2, k[d(AAu, Au) + 0]/2\}\\ &= d(AAu, Au),\end{aligned}$$

a contradiction. Thus $Au = AAu = SAu$, i.e., Au is a common fixed point of A and S.

Similarly, we can prove that $Bw = BBw$. Note $BBw = TBw$ and $Bw = Au$, hence $BAu = TAu = Au$, i. e., Au is also a common fixed point of B and T.

Now we prove the uniqueness of the common fixed point. If there exists another common fixed point v in X such that $v \neq Au$, then by (ii) we get

$$\begin{aligned} d(v, Au) &= d(Av, Bw) \\ &< \max\{d(Sv, Tw), k[d(Av, Sv) + d(Bw, Sv)]/2, k[d(Av, Tw) + d(Bw, Tw)]/2\} \\ &= d(Sv, Tw) = d(Av, Bw), \end{aligned}$$

a contradiction. Hence the common fixed point of A, B, S and T is unique.

The proof is similar when TX is assumed to be a complete subspace of X. The case in which AX or BX is a complete subspace of X is similar to the case in which TX or SX respectively is complete since $AX \subset TX$ and $BX \subset SX$. This completes the proof.

Theorem 1 is a revision of Theorem B. In Theorem 1, taking $A = B = f$ and $S = T = g$, we obtain a revision of Theorem A, i. e., the following theorem.

Theorem 2 Let f and g be noncompatible and pointwise R-weakly commuting selfmaps of a metric space (X, d) satisfying the conditions:

(1) $f(X) \subset g(X)$.

(2) $d(fx, fy) < \max\{d(gx, gy), k[d(fx, gx) + d(fy, gx)]/2, k[d(fx, gy) + d(fy, gy)]/2\}$, $1 \leqslant k < 2$, $x \neq y$.

If the range of f or g is a complete subspace of X then f and g have a unique common fixed point.

Remark 3 Pant[7] also found that Theorem 2.1 and 2.3 in [3] given by themselves were incorrect. They didn't give a counterexample, but give a modification which is different from our theorems.

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