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# Existence of Nontrivial Solution for Biharmonic Problems Involving Critical Sobolev Exponents

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**Abstract**: It is proved that existence of a nontrivial solution for biharmonic problem involving a critical Sobolev exponent

$$\begin{cases} \Delta^2 u = \mu \dfrac{u}{|x|^s} + |u|^{2^*-2}u + \lambda u + f(x), & x \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & x \in \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain and $0 \in \Omega$, $N \geqslant 5$, $0 \leqslant s \leqslant 4$, $0 \leqslant \mu < \bar{\mu} = \left(\dfrac{N(N-4)}{4}\right)^2$, $2^* = \dfrac{2N}{N-4}$ is the critical Sobolev exponent, $u$, $v$ is the outer normal vector on $\partial\Omega$, and $f(x)$ is a given function. By using the variational principle, we prove the existence of nontrivial solution for biharmonic problem involving the critical Sobolev exponent

**Key words**: biharmonic problem, critical Sobolev exponent, nontrivial solution
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# 含临界指数的双调和问题非平凡解的存在性

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[摘要] 讨论如下含临界指数的双调和方程非平凡解的存在性

$$\begin{cases} \Delta^2 u = \mu \dfrac{u}{|x|^s} + |u|^{2^*-2}u + \lambda u + f(x), & x \in \Omega, \\ u = \dfrac{\partial u}{\partial v} = 0, & x \in \partial\Omega. \end{cases}$$

其中 $\Omega \subset \mathbf{R}^N$ 是有界光滑区域, $0 \in \Omega$, $N \geqslant 5$, $0 \leqslant s \leqslant 4$, $0 \leqslant \mu < \bar{\mu} = \left(\dfrac{N(N-4)}{4}\right)^2$, $2^* = \dfrac{2N}{N-4}$为 $W^{2,2}(\Omega)$中 Sobolev嵌入的临界指数, $u$, $v$表示$\partial\Omega$ 的外法线方向, $f(x)$为给定函数. 通过变分方法, 我们证明了含临界指数的双调和方程非平凡解的存在性.

[关键词] 双调和方程, Sobolev临界指数, 非平凡解

## 0 Introduction

In this paper we will discuss the existence of a nontrivial solution for the following biharmonic problem involving Sobolev exponents

$$\begin{cases} \Delta^2 u = \mu \dfrac{u}{|x|^s} + |u|^{2^*-2}u + \lambda u + f(x), & x \in \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

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where $\Omega$ is a bounded domain in $\mathbf{R}^N$ with $N\geqslant 5$ and $\partial\Omega$ are sufficiently smooth, $0\leqslant\mu<\bar{\mu}=\left(\frac{N(N-4)}{4}\right)^2$, $0\leqslant s\leqslant 4$, $2^*=\frac{2N}{N-4}$ is the critical Sobolev exponent for the embedding $H_0^2(\mathbf{R}^N)\hookrightarrow L^{2^*}(\mathbf{R}^N)$. $H_0^2(\Omega)$ is the complete of $C_0^\infty(\Omega)$ respect to the norm $\|u\|$, $v$ is the outer normal vector on $\partial\Omega$.

We are interested in the existence of nontrivial solutions of (1). Becuase of the lack of the compactness of corresponding energy functional,

$$I(u)=\frac{1}{2}\int_\Omega(|\Delta u|^2-\lambda u^2)-\frac{\mu}{2}\int_\Omega\frac{|u|^2}{|x|^s}-\frac{1}{2^*}\int_\Omega|u|^{2^*}-\int_\Omega f(x)u,$$

for $u\in H_0^2(\Omega)$ and this leads to many interesting existence and non-existence phenomena.

We consider first the case of $f(x)=0$ where $\mu=\lambda=0$, namely the equation

$$\begin{cases}\Delta^2u=|u|^{2^*-2}u, & x\in\Omega,\\ u=\dfrac{\partial u}{\partial v}=0, & x\in\partial\Omega.\end{cases}\tag{2}$$

It is well known that (2) admits no positive solutions if $\Omega$ is star shaped[1,2]. By the Pohozaev identity with the unique continuation property. This suggests that in order to obtain existence results for (2), one should either add subcritical perturbations or modify the topology of the geometry of the domain.

Ghoussoub and Yuan[3] studied the existence of a multiplicity of the solution for the following quasilinear PDE:

$$\begin{cases}-\Delta_p u=\lambda|u|^{r-2}u+\mu\dfrac{|u|^{q-2}u}{|x|^s}, & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases}\tag{3}$$

In this paper, our main interests are the problem (1) suggested by Ghoussoub and Yuan. For Kang and Deng[4] discussed the result of nontrivial solution when $q=2$ and $0\leqslant s\leqslant 2$.

Our main methods follow that of Brezis[5] where the best Sobolev embedding constant plays a important role, the corresponding results for biharmonic operator where established in [6-8].

Throughout this paper, we denote the equivalent norms of $u$ in $H_0^2(\Omega)$ and $L^p(\Omega)$ with $\|u\|=\left(\int_\Omega|\Delta u|^2-\mu\frac{|u|^2}{|x|^s}\right)^{\frac{1}{2}}$ and $|u|_p=\left(\int_\Omega|u|^p\right)^{\frac{1}{p}}$ respectively.

It is true that the weak solutions of problem (1) is equivalent to the nonzero critical points of the functional defined on $H_0^2(\Omega)$

$$I(u)=\frac{1}{2}\int_\Omega(|\Delta u|^2-\lambda u^2)-\frac{\mu}{2}\int_\Omega\frac{|u|^2}{|x|^s}-\frac{1}{2^*}\int_\Omega|u|^{2^*}-\int_\Omega f(x)u\tag{4}$$

As $2^*$ is the critical Sobolev exponent corresponding to the noncompact embedding of $H_0^2(\Omega)$ into $L^{2^*}(\Omega)$. $I(u)$ does not in general satisfy the Palais-Smail condition and it is not possible to obtain critical points of $I(u)$ via simple variational arguments which are based on the compactness of the Sobolev embedding.

Definition

$$\lambda_1(\mu)=\inf_{\substack{u\in H_0^2(\Omega)\\ |u|\neq 1}}\frac{\int_\Omega\left(|\Delta u|^2-\mu\dfrac{|u|^2}{|x|^s}\right)}{\int_\Omega|u|^2}\tag{5}$$

when $\mu=0$, $\lambda_1(0)$ is denoted by $S$. The results of this paper are coming by this theorem.

**Theorem** If $0<\lambda<\lambda_1(\mu)$, $N\geqslant 5$, and $0\leqslant\mu<\bar{\mu}$, $0\leqslant s\leqslant 4$, $f(x)\in \bar{H}_0^2(\Omega)$ and $f(x)\neq 0$. Assume that for

$$C_N=\left(\frac{8}{N-4}\right)^{\frac{N+12}{8}}$$

and for any $u$:

$$u\in H_0^2(\Omega),\ \int_\Omega|u|^{2^*}dx=1.$$

The following inequality holds:

$$\int_\Omega fu < C_N\{\|u\|^2 - \lambda|u|_2^2\}^{\frac{N+4}{8}}. \tag{6}$$

Then the problem (1) has at least one weak solution in $H_0^2(\Omega)$.

# 1 Preliminary Results

At first we assume that

$$J(u) = \|u\|_2^2 - \lambda|u|_2^2 - (2^* - 1)|u|_{2^*}^{2^*},\ u\in H_0^2(\Omega),$$

$$\Lambda := \left\{u\in H_0^2(\Omega) \mid \int_\Omega(|\Delta u|^2 - \lambda u^2) - \int_\Omega|u|^{2^*} - \int_\Omega fu - \mu\int_\Omega\frac{u^2}{|x|^s} = 0\right\},$$

$$\Lambda^+ := \{u\in\Lambda; J(u) > 0\},\ \Lambda^0 := \{u\in\Lambda; J(u) = 0\},$$

$$\Lambda^- := \{u\in\Lambda; J(u) < 0\},\ c_0 := \inf_{u\in\Lambda} I(u).$$

Before proving the theorem, we will prove several lemmas. First we prove a Sobolev inequality and a compactness result.

**Lemma 1.1** (Sobolev-Hardy inequality) For all $u\in H_0^2(\Omega)$, when $0\leqslant s\leqslant 4$, we have

$$\int_\Omega\frac{u^2}{|x|^s} \leqslant C\int_\Omega|\Delta u|^2.$$

**Proof** For $s=0$ or $s=4$, this is just the Sobolev-Hardy inequality. Since $2\leqslant 2^*(s) = \dfrac{2(n-s)}{n-4}\leqslant 2^*$, we have $0\leqslant s\leqslant 4$, we can therefore only consider the case of $0<s<4$. By the Hardy, Sobolev and Holder inequalities, we have

$$\int_\Omega\frac{|u|^2}{|x|^s} \leqslant \left(\int_\Omega|u|^{2^*}\right)^{\frac{N-4}{N}}\left(\int_\Omega|x|^{\frac{-sN}{4}}\right)^{\frac{4}{N}} \leqslant C\int_\Omega|\Delta u|^2\left(\int_\Omega|x|^{\frac{-sN}{4}}\right)^{\frac{4}{N}} \leqslant C\int_\Omega|\Delta u|^2\left(\int_{B(0,R)}|x|^{\frac{-sN}{4}}\right)^{\frac{4}{N}} =$$

$$C\omega^{\frac{4}{N}}\left(\int_0^R r^{N-1-\frac{sN}{4}}\right)^{\frac{4}{N}}\int_\Omega|\Delta u|^2 = C\omega^{\frac{4}{N}}R^{4-s}\int_\Omega|\Delta u|^2 = C\int_\Omega|\Delta u|^2.$$

**Lemma 1.2** Assume that $\lambda\in(0,\lambda_1(\mu))$ and $f\neq 0$ satisfies condition (6). Then for any $u\in H_0^2(\Omega)$, $u\neq 0$, there exists a unique $t^+ = t^+(u) > 0$ such that $t^+(u)u\in\Lambda^-$,

$$t^+ > \left|\frac{\int_\Omega(|\Delta u|^2 - \lambda u^2) - \mu\int_\Omega\frac{|u|^2}{|x|^s}}{(2^* - 1)\int_\Omega|u|^{2^*}}\right|^{\frac{1}{2^*-2}} = t_{\max},$$

and

$$I(t^+u) = \max_{t\geqslant t_{\max}} I(tu).$$

Moreover, if $\int_\Omega fu > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that

$$t^-(u)u\in\Lambda^+.$$

In particular,

$$t^- < \left|\frac{\int_\Omega(|\Delta u|^2 - \lambda u^2) - \mu\int_\Omega\frac{u^2}{|x|^s}}{(2^* - 1)\int_\Omega|u|^{2^*}}\right|^{\frac{1}{2^*-2}} = t_{\max}$$

and

$$I(t^-u) = \min_{0\leqslant t\leqslant t_{\max}} I(tu).$$

**Proof** For any $u\in H_0^2(\Omega)$, set

$$\Phi(t) = t\int_\Omega(|\Delta u|^2 - \lambda u^2) - t\mu\int_\Omega\frac{|u|^2}{|x|^s} - t^{2^*-1}\int_\Omega|u|^{2^*} - \int_\Omega fu.$$

From (6) and the fact that $\lambda\in(0,\lambda_1(\mu))$, we can deduce that $t = t_{\max}$ is the unique critical point of $\Phi(t)$ and $\Phi(t_{\max}) > 0$. By the fact that $2^* > 2$ and the properties of function $\Phi(t)$ we can calculate the conclusions of our lemma easily.

**Lemma 1.3** Assume that the condition (6) holds. Then for any $u\in\Lambda$, $u\neq 0$, we have

$$\|u\|^2-\lambda|u|_2^2-(2^*-1)\int_\Omega|u|^{2^*}\neq 0. \tag{7}$$

**Proof** For every $u\in\lambda$, we have

$$\int_\Omega(|\Delta u|^2-\lambda u^2)-\mu\int_\Omega\frac{|u|^2}{|x|^s}=\int_\Omega|u|^{2^*}+\int_\Omega fu. \tag{8}$$

If (7) doesn't hold, then there exists some $u_0\in\Lambda$, $u_0\neq 0$ such that

$$\|u_0\|^2-\lambda|u_0|_2^2-(2^*-1)|u_0|_{2^*}^{2^*}=0 \tag{9}$$

From (8) and (9) we deduce

$$\int_\Omega|u_0|^{2^*}=\frac{1}{2^*-2}\int_\Omega fu_0, \tag{10}$$

$$\int_\Omega(|\Delta u_0|^2-\lambda u_0^2)-\mu\int_\Omega\frac{|u_0|^2}{|x|^s}=\frac{2^*-1}{2^*-2}\int_\Omega fu_0. \tag{11}$$

Putting

$$v=\frac{u_0}{\left(\int_\Omega|u_0|^{2^*}\right)^{\frac{1}{2^*}}}.$$

From (10) and (11) we have

$$|u_0|_{2^*}^{2^*-1}=\frac{1}{2^*-2}\int_\Omega fv,$$

$$\frac{2^*-2}{2^*-1}\left(\frac{1}{2^*-2}\int_\Omega fv\right)^{\frac{1}{2^*-1}}\left(\int_\Omega(|\Delta v|^2-\lambda v^2)-\mu\int_\Omega\frac{|v|^2}{|x|^s}\right)=\int_\Omega fv$$

Thus

$$\frac{1}{2^*-1}(2^*-2)^{\frac{8}{N+4}}\int_\Omega\left(|\Delta v|^2-\lambda|v|^2-\mu\frac{|v|^2}{|x|^s}\right)=\left(\int_\Omega fv\right)^{\frac{8}{N+4}},$$

which gives

$$\int_\Omega fv=\left(\frac{1}{2^*-1}\right)^{\frac{N+4}{8}}(2^*-2)\left(\int_\Omega(|\Delta v|^2-\lambda v^2)-\mu\int_\Omega\frac{|v|^2}{|x|^s}\right)^{\frac{N+4}{8}},\quad |v|_{2^*}=1$$

which contradicts the assumption (6). Thus we completes the proof of this lemma

## 2 Proof of Theorem

In this section, we give the proof of the theorem. First, research the following Lemma 2.1.

**Lemma 2.1** Let $\varphi$ be a $C^2$ functional on a Hilbert space $H$ that is coercive and bounded below on the set $M=\{u\in H;\ u\neq 0$ and $\psi(u)=\langle\varphi'(u),u\rangle=0\}$. Suppose $\langle\psi'(u),u\rangle\neq 0$ for any $u\in M$ and for any $(u_n)$ in $M$ that is minimizing sequence for $\varphi$ on $M$, we have that $(\psi(u_n))$ is bounded in $H$ and

$$\lim_{n\to\infty}\inf|\langle\psi'(u_n),u_n\rangle|>0$$

Then for every minimizing sequence $(v_n)$ in $M$ for $\varphi$, there exists $(u_n)$ in $M$ such that $\varphi(u_n)\leqslant\varphi(v_n)$, $\lim_{n\to\infty}\|u_n-v_n\|=0$ and $\lim_{n\to\infty}\|\varphi'(u_n)\|=0$

In particular, if $\varphi$ verifies $(PS)_{M,C}$ where $C=\inf\varphi(M)$, then the set $K_c=\{u\in H;\ \varphi(u)=c,\ \varphi'(u)=0\}$ is not empty.

**Proof of Theorem** We verify that the set $\Lambda$ satisfies the hypothesis of Lemma 2.1. Suppose $(u_n)$ is a minimizing sequence for $I$ in $\Lambda$, since $I$ is coercive, we can assume $(u_n)$ is uniformly bounded. Moreover, since $I(0)=0$ and $c_0<0$, we can assume, modulo passing to a subsequence, that for some $\delta>0$, we have that $\|u\|\geqslant\delta$. Suppose now that

$$J(u_n)=\|u_n\|^2-\lambda|u_n|_2^2-(2^*-1)\int_\Omega|u_n|^{2^*}=o(1), \tag{12}$$

then for some constant $\gamma>0$, we get $\int_\Omega|u_n|^{2^*}\geqslant\gamma$ and

$$\left(\frac{\|u_n\|^2 - \lambda|u_n|_2^2}{2^* - 1}\right)^{\frac{N+4}{8}} - \left(\int_\Omega |u_n|^{2^*}\right)^{\frac{N+4}{8}} = o(1).$$

By the fact that $u_n \in \Lambda$ we get

$$\int_\Omega fu_n = (2^* - 2)\int_\Omega |u_n|^{2^*} + o(1).$$

Thus we have

$$0 < \mu_0 \gamma^{\frac{2^*}{2^*-2}} \leqslant (2^* - 2)\left(\frac{\|u_n\|^2 - \lambda|u_n|_2^2}{2^* - 1}\right)^{\frac{N+4}{8}} - \left(\int_\Omega |u_n|^{2^*}\right)^{\frac{N+4}{8}} = o(1),$$

which is a contradiction, thus (12) does not hold. Thus we have that

$$\lim_{n\to\infty}\inf|J(u_n)| > 0$$

Next, any $(u_n) \subset \Lambda$ such that $\lim_{n\to\infty} I(u_n) = c_o$ and $\lim_{n\to\infty} |I'(u_n)| = 0$, then $(u_n)$ is necessarily bounded and we can find a weak cluster point $u \in H_0^2(\Omega)$. We can also assume that $u_n \rightharpoonup u$ weakly in $H_0^2(\Omega)$.

In particular, for any $v \in H_0^2(\Omega)$,

$$< I'(u_n), v > = \int_\Omega \triangle u_n \triangle v - \mu \frac{u_n v}{|x|^s} - \lambda u_n v - fv - |u_n|^{2^*-2} u_n v,$$

as $n \to \infty$, we get

$$< I'(u), v > = \int_\Omega \triangle u \triangle v - \mu \frac{uv}{|x|^s} - \lambda uv - fv - |u|^{2^*-2} uv = 0$$

Hence $< I'(u), v > = 0$ for all $v \in H_0^2(\Omega)$, which means that $u$ is a nontrivial solution for (1), in particular, $u \in \Lambda$. Since $I$ is weakly lower semi-continuous, we also get

$$c_o \leqslant I(u) \leqslant \lim_{n\to\infty} I(u_n) = c_o,$$

thus $I(u) = c_o$ and $\|u_n\| \to \|u\|$ which implies that $u_n \to u$ strongly in $H_0^2(\Omega)$. The proof of Theorem is completed.

## 3 Conclusion

In this work we present the existence of nontrivial solution for biharmonic problems involving critical exponents, then we make some preliminary results of some lemmas using Sobolev Hardy and Holder inequalities which are used in the proof of the theorem, and then we find the problem (1) has at least one weak solution in $H_0^2(\Omega)$.

**[References]**

[1] Mitidieri E. A Rellich type identity and applications[J]. Comm P D E, 1993, 18: 125-151.

[2] Oswald P. On a priori estimates for positive solutions of a semilinear biharmonic equation in a ball[J]. Comment Math, Carolinae, 1985, 26: 565-577.

[3] Ghoussoub N, Yuan C. Multiplic solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents[J]. Trans Amer Math Soc, 2000, 352(12): 5703-5743.

[4] Kang K D, Deng Y B. Sobolev-Hardy inequalities and critical biharmonic problems[J]. Acta Math Sci, 2003, 23(A)(1): 106-114.

[5] Brezis H, Nirenberg L. Positive solutions of nonlinear elliptic equation involving critical Sobolev exponents[J]. Comm Pure Appl Math, 1983, 36: 436-477.

[6] Reng Y B, Yang J F. Existence of multiple solutions and bifurcation for critical semilinear biharmonic equations[J]. Sys Sci and Math Sci, 1995, 8(4): 319-326.

[7] Xuan B J, Chen Z C. Existence, multiplicity and bifurcation for critical polyharmonic equations[J]. Sys Sci and Math Sci, 1999, 12(1): 59-69.

[8] Yao Y X, Shen Y T, Chen Z H. Biharmonic equation and improved hardy inequality[J]. Acta Math Appl Sinica, 2004, 20(3): 433-440.

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