A Complete Classification of (12, 4, 1)-PMDs With a β_1-Block

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Abstract A perfect Mendelsohn design, denoted by (v, k, λ)-PMD is a pair $(X, \mathscr{A})$, where X is a v-set (of points), and $\mathscr{A}$ is a collection of cyclically ordered k-subsets of X (called blocks), such that every ordered pair of points of X appears t-apart in exactly λ blocks of $\mathscr{A}$ for any t, where $1 \leqslant t \leqslant k-1$. Let A be a block of $\mathscr{A}$, if there are exactly u blocks of $\mathscr{A}$, which have no common elements with A, then we say A is a β_u-block. In this article, it is proved that there are exactly 141 non-isomorphic (12, 4, 1)-PMDs with a β_1-block.

Key words β_u-block, perfect Mendelsohn design, isomorphism

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具有 β_1 型区组的(12, 4, 1)-PMD的完全分类

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[摘要] 设 X 是一个 v元点集, $\mathscr{A}$是循环有序的 k元子集簇. 一个完全 Mendelsohn设计, 记为 (v, k, λ)-PMD, 是二元组 $(X, \mathscr{A})$, 使得 X 中每个有序点对恰好 t间隔地出现在 λ个区组中. 若一个区组恰有 u个区组与之不交, 则称之为 β_u 区组. 本文证明了共有 141个不同构具有 β_1 型区组的 (12, 4, 1)-PMD.

[关键词] β_u-区组, 完全, Mendelsohn设计, 同构

0 Introduction

Let v, k, λ and n be positive integers. $(x_1, x_2, \cdots, x_k)$ is defined to be $\{(x_i, x_j): i \neq j, i, j = 1, 2, \cdots, k\}$, in which the ordered pair (x_i, x_j) is called $(j-i)$-apart for $i<j$ and $(k+j-i)$-apart for $i>j$, and is called a cyclically ordered k-subset of $\{x_1, x_2, \cdots, x_k\}$.

A perfect Mendelsohn design, denoted by (v, k, λ)-PMD is a pair $(X, \mathscr{A})$, where X is a v-set (of points), and $\mathscr{A}$ is a collection of cyclically ordered k-subsets of X (called blocks), such that every ordered pair of points of X appears t-apart in exactly λ blocks of $\mathscr{A}$ for any t, where $1 \leqslant t \leqslant k-1$.

For the existence problem for $(v, 4, 1)$-PMD, Mendelson[1] first investigated the existence of $(v, 4, 1)$-PMDs by associating these designs with a variety of quasigroups. However, there was no concerted effort made to determine the spectrum of these designs until an almost complete solution was given for the case $v \equiv 1 \pmod 4$ in Bennet[2], by exploiting PBD-closure. Zhang Xuebin[3] had considerable success with the case $v \equiv 0 \pmod 4$. Subsequent investigations culminated in the following conclusive result for $(v, 4, \lambda)$ in Bennett et al[4]. It

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should be remarked that the most outstanding case for the existence of a (12, 4, 1)-PMD was finally settled in a computer search for its associated quasigroup and the example was provided in Bennett et al.[5]. Hence we have

Theorem 0.1 A $(v, 4, 1)$-PMD exists if and only if $v(v-1)\equiv 0 \pmod 4$ with the exception of $v=4$ and $v=8$.

Definition 0.1 Let $(X, \mathscr{A})$ be a (v, k, λ)-PMD. Let A be a block of $\mathscr{A}$. If there are exactly u blocks of $\mathscr{A}$, which have no common elements with A, then we say A is a β_u-block.

Lemma 0.1 Let $(X, \mathscr{A})$ be a (v, k, λ)-PMD and $\mathscr{A}^{-1}=\{(d, c, b, a): (a, b, c, d)\in\mathscr{A}\}$. Then $(X, \mathscr{A}^{-1})$ is also a (v, k, λ)-PMD.

Definition 0.2 Let $\mathscr{A}$ and $\mathscr{C}$ be two collections of cyclically ordered k-subsets of X. We say they are isomorphic if

(i) there exists a bijection α from X to X such that $\alpha(\mathscr{A})=\mathscr{C}$;

(ii) $\mathscr{C}=\mathscr{A}^{-1}$.

Let $(X, \mathscr{A})$ and $(X, \mathscr{C})$ be two (v, k, λ)-PMDs. We say they are isomorphic if $\mathscr{A}$ and $\mathscr{C}$ are isomorphic.

Lemma 0.2 Let $(X, \mathscr{A})$ and $(X, \mathscr{C})$ be two (v, k, λ)-PMDs. If α is an isomorphic mapping and A is a β_u-block of $\mathscr{A}$, then $\alpha(A)$ is a β_u-block of $\mathscr{C}$.

From Lemma 0.2 we have

Lemma 0.3 Let $(X, \mathscr{A})$ and $(X, \mathscr{C})$ be two (v, k, λ)-PMDs. If there is an integer u such that the number of β_u-blocks of $\mathscr{A}$ is not equal to that of $\mathscr{C}$, then they are not isomorphic.

In this article, we will show that there are exactly 141 non-isomorphic (12, 4, 1)-PMDs with a β_1-block.

1 Intersecting Equations

A (v, k, λ) balanced incomplete block design (briefly BIBD) is a pair $(X, \mathscr{B})$ where X is a v-set of elements called points, and $\mathscr{B}$ is a set of k-subsets of X called blocks with the property that every pair of X is contained in exactly λ blocks.

Lemma 1.1 Let $(X, \mathscr{A})$ be a (12, 4, 1)-PMD and $\mathscr{B}=\{\{a, b, c, d\}: (a, b, c, d)\in\mathscr{A}\}$. Then $(X, \mathscr{B})$ is a (12, 4, 3)-BIBD with no repeated blocks.

Lemma 1.2 Let $(X, \mathscr{A})$ be a (12, 4, 1)-PMD. Let A be a block of $\mathscr{A}$. Let x_i denote the number of the blocks of $\mathscr{A}$ which have exactly i common elements with A. Then we have the following equations:

$$x_0+x_1+x_2+x_3+x_4=33,\ x_1+2x_2+3x_3+4x_4=44,\ x_2+3x_3+6x_4=18,\ x_4=1$$

and their solutions are: $(x_0, x_1, x_2, x_3, x_4)\in\{(0, 28, 0, 4, 1), (1, 25, 3, 3, 1), (2, 22, 6, 2, 1), (3, 19, 9, 1, 1), (4, 16, 12, 0, 1)\}$.

From Lemma 1.2, we have

Lemma 1.3 Let $(X, \mathscr{A})$ be a (12, 4, 1)-PMD and A be a β_k-block of $\mathscr{A}$. Then $k=0, 1, 2, 3, 4$.

Lemma 1.4 Let $(X, \mathscr{A})$ be a (12, 4, 1)-PMD. Let A be a block of $\mathscr{A}$. Let $z\notin A$, and c_i denote the number of the blocks of $\mathscr{A}$ which have z and exactly i elements of A. Then we have the following equations:

$$c_0+c_1+c_2+c_3=11,\ c_1+2c_2+3c_3=12$$

From Lemma 1.2, $c_0+c_3\leqslant 4$, so we have

Lemma 1.5 The solutions for the equations in Lemma 1.4 are $(c_0, c_1, c_2, c_3)\in\{(4, 2, 5, 0), (3, 4, 4, 0), (2, 6, 3, 0), (1, 8, 2, 0), (0, 10, 1, 0), (3, 5, 2, 1), (2, 7, 1, 1), (1, 9, 0, 1)\}$.

2 Structure for (12, 4, 1)-PMD With a β_1-Block

Lemma 2.1 Let $(X, \mathscr{A})$ be a (12, 4, 1)-PMD and A be a β_k-block of $\mathscr{A}$. Then $k=1, 2, 3, 4$.

Proof From Lemma 1.3, $k=0, 1, 2, 3, 4$. Assume A is a β_0-block and $z\notin A$. It follows from Lemma 1.2 that $x_0=x_2=0$, this forces that $c_0=c_2=0$. Hence by Lemma 1.4, we have $2c_3=1$, which has no integer solutions.

For $A=(a,b,c,d)$ we difine $\pi(A)=\{a,b,c,d\}$. Let

$B_0=\{3,6,9,12\}$, $B_1=\{1,4,7,10\}$, $B_2=\{1,3,6,9\}$, $B_3=\{4,6,9,12\}$,

$B_4=\{7,3,9,12\}$, $B_5=\{10,6,12,2\}$, $B_6=\{10,3,12,5\}$, $B_7=\{11,3,6,8\}$.

Theorem 2.1 Let $(X,\mathscr{A})$ be a $(12,4,1)$-PMD with a β_1 block A_0. Then there is an isomorphism α and seven blocks A_i, $i=1,2,\cdots,7$, such that $\pi(\alpha(A_k))=B_k$, $k=0,1,2,\cdots,7$.

Proof Let $\pi(\alpha(A_0))=\{3,6,9,12\}$. Since A_0 is a β_1 block, $\alpha(A_0)$ is also a β_1 block. By Lemma 1.2, we can let

$\pi(\alpha(A_0))=\{3,6,9,12\}$, $\pi(\alpha(A_1))=\{1,4,7,10\}$, $\pi(\alpha(A_2))=\{3,6,9,-\}$,

$\pi(\alpha(A_3))=\{6,9,12,-\}$, $\pi(\alpha(A_4))=\{3,9,12,-\}$, $\pi(\alpha(A_5))=\{6,12,-,-\}$,

$\pi(\alpha(A_6))=\{3,12,-,-\}$, $\pi(\alpha(A_7))=\{3,6,-,-\}$.

Apply Lemma 1.5 with $z\in\{1,4,7,10\}$ and $\alpha(A_0)$, it is easy to see that $c_0=1$, $c_1=9$, $c_2=0$, $c_3=1$ or $c_0=1$, $c_1=8$, $c_2=2$, $c_3=0$. Hence we have

$\pi(\alpha(A_2))=\{1,3,6,9\}$, $\pi(\alpha(A_3))=\{4,6,9,12\}$, $\pi(\alpha(A_4))=\{7,3,9,12\}$,

$\pi(\alpha(A_5))=\{10,6,12,-\}$, $\pi(\alpha(A_6))=\{10,3,12,-\}$, $\pi(\alpha(A_7))=\{3,6,-,-\}$.

Apply Lemma 1.5 with $z\in\{2,5,8,11\}$ and $\alpha(A_0)$, it is easy to see that $c_0=0$, $c_1=10$, $c_2=1$, $c_3=0$. Hence we have

$$\pi(\alpha(A_5))=\{10,6,12,2\},\ \pi(\alpha(A_6))=\{10,3,12,5\},\ \pi(\alpha(A_7))=\{11,3,6,8\}.$$

Let

$R_1=(3,6,9,12)$, $R_2=(3,12,6,9)$, $S_1=(10,1,4,7)$, $S_2=(10,1,7,4)$,

$S_3=(10,4,1,7)$, $S_4=(10,4,7,1)$, $S_5=(10,7,1,4)$, $S_6=(10,7,4,1)$.

$\mathscr{U}_1=\{(1,3,9,6),(4,6,12,9),(7,9,3,12)\}$,

$\mathscr{T}_1=\{R_1\}\cup\mathscr{U}_1\cup\{(11,8,6,3),(10,2,12,6),(10,3,5,12)\}$,

$\mathscr{T}_2=\{R_1\}\cup\mathscr{U}_1\cup\{(11,8,6,3),(10,2,12,6),(10,12,5,3)\}$,

$\mathscr{T}_3=\{R_1\}\cup\mathscr{U}_1\cup\{(11,8,6,3),(10,12,6,2),(10,3,5,12)\}$,

$\mathscr{U}_2=\{(1,6,3,9),(4,9,6,12),(7,12,9,3)\}$,

$\mathscr{T}_4=\{R_1\}\cup\mathscr{U}_2\cup\{(11,3,8,6),(10,2,12,6),(10,5,3,12)\}$,

$\mathscr{T}_5=\{R_1\}\cup\mathscr{U}_2\cup\{(11,3,8,6),(10,12,6,2),(10,5,3,12)\}$,

$\mathscr{T}_6=\{R_1\}\cup\mathscr{U}_2\cup\{(11,3,8,6),(10,12,6,2),(10,3,12,5)\}$,

$\mathscr{V}=\{(4,12,9,6),(1,6,3,9),(7,9,12,3)\}$,

$\mathscr{T}_7=\{R_2\}\cup\mathscr{V}\cup\{(11,8,3,6),(10,2,6,12),(10,12,5,3)\}$,

$\mathscr{T}_8=\{R_2\}\cup\mathscr{V}\cup\{(11,8,3,6),(10,6,12,2),(10,3,5,12)\}$,

$\mathscr{T}_9=\{R_2\}\cup\mathscr{V}\cup\{(11,8,3,6),(10,6,12,2),(10,12,5,3)\}$.

By Definition 0.2 we have

Lemma 2.2 If $(i,j)\neq(m,n)$, then $\{S_i\}\cup\mathscr{T}_j$ and $\{S_m\}\cup\mathscr{T}_n$ are not isomorphic.

By Definition 0.2 and Theorem 2.1 we have

Lemma 2.3 There are integers i and j such that

$\{\alpha(A_k): k=0,1,\cdots,7\}$ (in Theorem 2.1) and $\{S_i\}\cup\mathscr{T}_j$ are isomorphic.

3 Non-isomorphic (12, 4, 1)-PMDs

In the following we will show that it is easy to find all non-isomorphic $(12,4,1)$-PMDs with $\{S_i\}\cup\mathscr{T}_j$.

Example 3.1 Find all non-isomorphic $(12,4,1)$-PMDs with $\{S_1\}\cup\mathscr{T}_1$. Since

$$\{S_1\}\cup\mathscr{T}_1=\{(3,6,9,12),(10,1,4,7),(1,3,9,6),(4,6,12,9),$$
$$(7,9,3,12),(10,2,12,6),(10,3,5,12),(11,8,6,3)\}.$$

It is easy to see that the other 25 blocks should be

(3, 1, –, –), (3, 2, –, –), (3, 4, –, –), (3, 7, –, –), (3, 8, –, –),
(3, 10, –, –), (6, 2, –, –), (6, 4, –, –), (6, 5, –, –), (6, 7, –, –),
(6, 8, –, –), (6, 11, –, –), (9, 1, –, –), (9, 2, –, –), (9, 5, –, –),
(9, 7, –, –), (9, 8, –, –), (9, 10, –, –), (9, 11, –, –), (12, 1, –, –),
(12, 2, –, –), (12, 4, –, –), (12, 5, –, –), (12, 8, –, –), (12, 11, –, –).

With the help of computer, there are three (12, 4, 1)-PMDs as follows

(i) $\mathscr{C}$ = {(3, 6, 9, 12), (10, 1, 4, 7), (1, 3, 9, 6), (4, 6, 12, 9), (7, 9, 3, 12), (10, 2, 12, 6), (10, 3, 5, 12), (11, 8, 6, 3), (3, 1, 2, 4), (3, 2, 10, 7), (3, 4, 5, 8), (3, 7, 1, 5), (3, 8, 11, 2), (3, 10, 4, 11), (6, 2, 5, 11), (6, 4, 10, 5), (6, 5, 7, 2), (6, 7, 8, 10), (6, 8, 4, 1), (6, 11, 1, 7), (9, 1, 10, 11), (9, 2, 11, 4), (9, 5, 2, 1), (9, 7, 4, 8), (9, 8, 5, 10), (9, 10, 8, 2), (9, 11, 7, 5), (12, 1, 11, 10), (12, 2, 8, 1), (12, 4, 2, 7), (12, 5, 1, 8), (12, 8, 7, 11), (12, 11, 5, 4)}.

(ii) $\mathscr{C}$ = {(3, 6, 9, 12), (10, 1, 4, 7), (1, 3, 9, 6), (4, 6, 12, 9), (7, 9, 3, 12), (10, 2, 12, 6), (10, 3, 5, 12), (11, 8, 6, 3), (3, 1, 5, 4), (3, 2, 10, 8), (3, 4, 2, 5), (3, 7, 11, 2), (3, 8, 1, 7), (3, 10, 4, 11), (6, 2, 4, 5), (6, 4, 1, 11), (6, 5, 10, 7), (6, 7, 8, 10), (6, 8, 5, 1), (6, 11, 7, 2), (9, 1, 8, 2), (9, 2, 7, 4), (9, 5, 2, 8), (9, 7, 5, 11), (9, 8, 4, 10), (9, 10, 11, 1), (9, 11, 10, 5), (12, 1, 2, 11), (12, 2, 1, 10), (12, 4, 8, 7), (12, 5, 7, 1), (12, 8, 11, 4), (12, 11, 5, 8)}.

(iii) $\mathscr{C}$ = {(3, 6, 9, 12), (10, 1, 4, 7), (1, 3, 9, 6), (4, 6, 12, 9), (7, 9, 3, 12), (10, 2, 12, 6), (10, 3, 5, 12), (11, 8, 6, 3), (3, 1, 11, 2), (3, 2, 10, 11), (3, 4, 1, 5), (3, 7, 2, 4), (3, 8, 5, 7), (3, 10, 4, 8), (6, 2, 7, 5), (6, 4, 5, 1), (6, 5, 10, 7), (6, 7, 8, 11), (6, 8, 1, 2), (6, 11, 4, 10), (9, 1, 10, 8), (9, 2, 8, 4), (9, 5, 4, 11), (9, 7, 11, 10), (9, 8, 2, 5), (9, 10, 5, 2), (9, 11, 7, 1), (12, 1, 8, 10), (12, 2, 1, 7), (12, 4, 2, 11), (12, 5, 11, 1), (12, 8, 7, 4), (12, 11, 5, 8)}.

It is easily checked that for (i) there are 1 β_1-block, no β_2-block, 13 β_3-blocks and 19 β_4-blocks. We also say (1, 0, 13, 19) is the type vector of (i). It is also easily checked that the type vector of (ii) is (1, 4, 17, 11), and the type vector of (iii) is (1, 1, 19, 12). Hence (i), (ii) and (iii) are three non-isomorphic (12, 4, 1)-PMDs by Lemma 0.3.

In the same way, we have the following theorem.

Lemma 3.1 There are at most 144 non-isomorphic (12, 4, 1)-PMDs with a β_1-block.

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