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Cascadic Multigrid Method for Mortar-Type Rotated Q_1 Element

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Abstract: In this paper, the cascadic multigrid method for mortar-type rotated Q_1 element is discussed. It is proved that the cascadic conjugate gradinet method is optimal and the cascadic multigrid method with traditional iteration is nearly optimal. Numerical results confirm our theoretical analysis.

Key words: mortar element, rotated Q_1 element, cascadic multigrid method

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Mortar型旋转 Q_1 元的瀑布型多重网格方法

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[摘要]　展现了 mortar型旋转 Q_1 元的瀑布型多重网格方法. 证明了采用共轭梯度作为光滑子的瀑布型多重网格法是最优的, 而采用其它传统迭代作光滑子的瀑布型多重网格法是拟最优的. 并通过数值试验验证了我们的理论结果.

[关键词]　mortar元, 旋转 Q_1 元, 瀑布型多重网格法

0　Introduction

The rotated Q_1 element is an important nonconforming quadrilateral element. It was first proposed and analyzed in [1] for Stokes problems. Recently there are many papers to deal with the so-called mortar element method[2-4]. In [3], a mortar-type rotated Q_1 element method was proposed, and the optimal error estimate in energy norm was obtained.

The cascadic multigrid method[5] is a new kind of multigrid method. Compared with usual multigrid methods, it requires no coarse grid correction at all and may be viewed as a "one way" multigrid method. The general framework to analyze the cascadic multigrid method was proposed in [6]. For second order elliptic problem discretized by mortar-type rotated Q_1 element, we proved that the W-cycle multigrid method is optimal, and that a variable V-cycle multigrid algorithm is presented[7]. In this paper, we consider the cascadic multigrid method for the discrete problem.

1　The Mortar-Type Rotated Q_1 Element Method

For simplicity, we consider the following model problem

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$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a rectangular domain, $f \in L^2(\Omega)$. There is no difficulty to extend the results in this paper to more general second order elliptic problems.

The variational form of (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{1.2}$$

where the bilinear form $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$. By assumption, it is well-known that for any $f \in L^2(\Omega)$ there is a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of (1.2), which satisfies $\|u\|_2 \lesssim \|f\|_0$ [8].

In this paper, the domain Ω is divided into N non-overlapping rectangular subdomains, $\overline{\Omega} = \bigcup_{k=1}^N \overline{\Omega}_k$, where $\overline{\Omega}_i \cap \overline{\Omega}_j$ is empty, or a vertex, or an edge for $i \neq j$. The interface $\Gamma = \bigcup_{k=1}^N \partial\Omega_k \setminus \partial\Omega$ is broken into a set of disjoint open straight segments γ_m $(1 \le m \le M)$ (that is the edges of subdomains) called mortars. We denote the common open edge to Ω_i and Ω_j by γ_m. By $\gamma_{m(i)}$ we denote an edge of Ω_i and call it mortar and by $\delta_{m(j)}$ an edge of Ω_j that geometrically occupies the same place as $\gamma_{m(i)}$ and is called nonmortar.

With each Ω_k we associate a quasi-uniform partition $\mathbf{T}_h^k$ made of elements that are rectangles whose edges are parallel to x-axis or y-axis. The mesh parameter h_k is the diameter of the largest element in $\mathbf{T}_h^k$. Let $\mathbf{T}_h = \bigcup_{k=1}^N \mathbf{T}_h^k$ with $h = \max_{1 \le k \le N} h_k$, and assume that

$$\exists\, \sigma > 0, \text{ such that } \frac{\max_{1 \le i \le N} h_i}{\min_{1 \le j \le N} h_j} \le \sigma. \tag{1.3}$$

For each partition $\mathbf{T}_h^k$, the rotated Q_1 element space is defined by

$$V_h^k(\Omega_k) = \{ v \in L^2(\Omega_k) \mid v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 (x^2 - y^2),\ a_E^i \in \mathbf{R},$$

$$\int_{\partial E \cap \partial\Omega} v|_{\partial\Omega} \mathrm{d}s = 0,\ \forall E \in \mathbf{T}_h^k(\Omega_k); \text{ for } E_1, E_2 \in \mathbf{T}_h^k(\Omega_k), \text{ if } \partial E_1 \cap \partial E_2 = e,$$

$$\text{then } \int_e v|_{\partial E_1} \mathrm{d}s = \int_e v|_{\partial E_2} \mathrm{d}s \}.$$

Let

$$\widetilde{V}_h = \prod_{k=1}^N V_h^k = \{ v_h \mid v_h^k = v_h|_{\Omega_k} \in V_h^k(\Omega_k) \}.$$

For any interface $\gamma_m = \gamma_{m(i)} = \delta_{m(j)}$, $1 \le m \le M$, there are two different and independent 1-D partitions $\mathbf{T}_h(\gamma_{m(i)})$ and $\mathbf{T}_h(\delta_{m(j)})$. An auxiliary test space $M_h(\delta_{m(j)})$ is defined by

$$M_h(\delta_{m(j)}) = \{ v \in L^2(\delta_{m(j)}) \mid v \text{ is piecewise constant on elements of the nonmortar partition } \mathbf{T}_h(\delta_{m(j)}) \}.$$

The dimension of $M_h(\delta_{m(j)})$ is equal to the number of elements on $\delta_{m(j)}$.

For each nonmortar edge $\delta_{m(j)}$, we define a L^2-orthogonal projection operator $Q_{h,\delta_{m(j)}}: L^2(\gamma_m) \to M_h(\delta_{m(j)})$ by

$$(Q_{h,\delta_{m(j)}} v, w)_{L^2(\delta_{m(j)})} = (v, w)_{L^2(\delta_{m(j)})}, \qquad \forall w \in M_h(\delta_{m(j)}), \tag{1.4}$$

where $(\cdot, \cdot)_{L^2(\delta_{m(j)})}$ denotes the L^2 inner product over the space $L^2(\delta_{m(j)})$.

For the projection operator $Q_{h,\delta_{m(j)}}$, we have

Lemma 1.1[3] If $u \in H^{1/2}(\delta_{m(j)})$, then

$$\| u - Q_{h,\delta_{m(j)}} u \|_{0,\delta_{m(j)}} \lesssim h_j^{1/2} |u|_{H^{1/2}(\delta_{m(j)})}.$$

We now define the following mortar-type rotated Q_1 element space

$$V_h = \{ v \in \widetilde{V}_h \mid Q_{h,\delta_{m(j)}}(v|_{\delta_{m(j)}}) = Q_{h,\delta_{m(j)}}(v|_{\gamma_{m(i)}}) \text{ for } \forall \gamma_{m(i)} = \delta_{m(j)} \subset \Gamma \}.$$

The condition of the equality of the L^2-orthogonal projection of traces onto the test space for each interface is called mortar condition.

Define

$$a_h^k(u_h^k, v_h^k) = \sum_{E\in T_h^k}\int_E \nabla u_h^k \cdot \nabla v_h^k \mathrm{d}x,\ \forall u_h^k, v_h^k \in V_h^k,\ \text{and}\ a_h(u_h, v_h) = \sum_{k=1}^N a_h^k(u_h, v_h),\ \forall u_h,\ v_h \in \widetilde{V}_h.$$

Then we denote

$$|v_h^k|_{h,k}^2 = a_h^k(v_h^k, v_h^k),\quad \forall v_h^k \in V_h^k,\ \text{and}\ |v_h|_h^2 = \sum_{k=1}^N |v_h|_{h,k}^2,\quad \forall v_h \in \widetilde{V}_h.$$

The mortar-type rotated Q_1 element approximation of problem (1.2) is to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h),\quad \forall v_h \in V_h. \tag{1.5}$$

Let $T_{h/2}^{\ k}$ be the partition which is constructed by connecting midpoints of the opposite edges of elements of $T_h^{\ k}$, $W_{h/2}^k(\Omega_k)$ be piecewise bilinear conforming element space defined on $T_{h/2}^{\ k}$.

As in [3], we introduce a local operator $\mathbf{M}_k: V_h^k(\Omega_k) \to W_{h/2}^k(\Omega_k)$, and has the following properties

Lemma 1.2[3] For any $v_h^k \in V_h^k(\Omega_k)$, we have

$$|v_h^k|_{h,k} \lesssim |\mathbf{M}_k v_h^k|_{h,k} \lesssim |v_h^k|_{h,k}, \tag{1.6}$$

$$\| v_h^k - \mathbf{M}_k v_h^k \|_{L^2(\varepsilon)} \lesssim h_k^{1/2} |v_h^k|_{h,k}, \tag{1.7}$$

where ε is an edge of Ω_k.

The following error estimate can be found in [3].

Theorem 1.1 Let u, u_h be the solutions of (1.2) and (1.5) respectively, then

$$|u - u_h|_h^2 \lesssim \sum_{k=1}^N h_k^2 |u|_{2,\Omega_k}^2. \tag{1.8}$$

2 Error Estimate in L^2-Norm

Consider the following auxiliary problem: find $\phi \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta\phi = g & \text{in}\ \Omega, \\ \phi = 0 & \text{on}\ \partial\Omega, \end{cases} \tag{2.1}$$

where $g \in L^2(\Omega)$. Obviously, the problem (2.1) also has the corresponding H^2-regularity

$$\|\phi\|_2 \lesssim \|g\|_0. \tag{2.2}$$

In order to get the error estimate in L^2-norm, we first give the following lemmas whose proof is similar to Lemma 3.3 and Lemma 3.4 in [9] separately.

Lemma 2.1 Assume that u, u_h and ϕ are the solutions of (1.2), (1.5) and (2.1) respectively. Then we have

$$\left| \sum_{k=1}^N \sum_{E\in T_h^k} \int_{\partial E} (u - u_h) \frac{\partial\phi}{\partial n} \mathrm{d}s \right| \lesssim h^2 \|u\|_2 \|g\|_0, \tag{2.3}$$

where n is the unit outerward normal vector along ∂E.

Lemma 2.2 Assume that u, u_h and ϕ are the solutions of (1.2), (1.5) and (2.1) respectively. Then we have

$$\left| \sum_{k=1}^N \sum_{E\in T_h^k} \int_{\partial E} \nabla(u - u_h) \cdot \nabla\phi\, \mathrm{d}x \right| \lesssim h^2 \|u\|_2 \|g\|_0. \tag{2.4}$$

Now, by Lemmas 2.1～2.2, we get the main result of this section.

Theorem 2.1 Let u, u_h be the solutions of (1.2) and (1.5) respectively. Then

$$\|u - u_h\|_0 \lesssim h^2 \|u\|_2. \tag{2.5}$$

3 Cascadic Multigrid Method

Let T_1 be the coarsest partition of Ω. We refine T_1 to produce T_2 by splitting each rectangle of T_1 into four rectangles by jointing the opposite midpoints of the edges of the rectangle. The partition T_2 is quasi-uniform of size $h_2 = h_1/2$. Repeating this process, we get a sequence of partitions T_l, $l = 1, 2, \cdots, L$, each quasi-uniform of size $h_l = h_1/2^{l-1}$, and denote V_l as the mortar-type rotated Q_1 element space over the partition $\mathbf{T}_l$. Obviously, we have $V_1 \not\subseteq V_2 \not\subseteq \cdots \not\subseteq V_L$.

If we change the index h in section 1 to be l, then the discrete problem of (1.2) on V_l is to find $u_l \in V_l$ such that

$$a_l(u_l, v_l) = (f, v_l), \quad \forall v_l \in V_l. \tag{3.1}$$

In this section, we apply the framework developed in [6] to prove the convergence of our cascadic multigrid. Before giving the algorithms, we define a suitable intergrid transfer operator for the nonnested spaces. First we define an operator $\mathbf{\Pi}_l^k: V_{l-1}^k \to V_l^k$, $\forall v^k \in V_{l-1}^k$ by

$$\frac{1}{|e|}\int_e \mathbf{\Pi}_l^k(v^k)\,ds = \begin{cases} 0 & e \subset \partial\Omega_k \cap \partial\Omega, \\ \frac{1}{|e|}\int_e v^k ds & e \subset \partial\Omega_k \setminus \partial\Omega, \\ \frac{1}{|e|}\int_e v^k ds & e \not\subset \partial E,\ E \in \mathbf{T}_{l-1}^k, \\ \frac{1}{2|e|}\int_e (v^k|_{E_1} + v^k|_{E_2})\,ds & e \subset \partial E_1 \cap \partial E_2,\ E_1, E_2 \in \mathbf{T}_{l-1}{}^k, \end{cases} \tag{3.2}$$

where $e \subset \partial E$, $E \in \mathbf{T}_l^k$.

Based on the operator $\mathbf{\Pi}_l^k$, we define an transfer operator $\mathbf{\Pi}_l: \widetilde{V}_{l-1} \to \widetilde{V}_l$ as follows

$$\mathbf{\Pi}_l v = (\mathbf{\Pi}_l^1 v^1, \mathbf{\Pi}_l^2 v^2, \cdots, \mathbf{\Pi}_l^N v^N), \quad \forall v = (v^1, v^2, \cdots, v^N) \in \widetilde{V}_{l-1}.$$

Define the operator $\Xi_{l,\delta_{m(j)}}: \widetilde{V}_l \to \widetilde{V}_l$ by

$$\int_e \Xi_{l,\delta_{m(j)}} v\,ds = \begin{cases} \int_e Q_{l,\delta_{m(j)}}(v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}})\,ds & e \in \mathbf{T}_l(\delta_{m(j)}), \\ 0 & \text{otherwise}, \end{cases} \tag{3.3}$$

where $e \subset \partial E$, $E \in \mathbf{T}_l$. Then for any $v \in \widetilde{V}_l$, set $v^* = v + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(v)$, we can check that $v^* \in V_l$ [7].

After the above preparation, we can define an intergrid transfer operator $I_l: \widetilde{V}_{l-1} \to V_l$. For any $v \in \widetilde{V}_{l-1}$, let

$$I_l v = \mathbf{\Pi}_l v + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(\mathbf{\Pi}_l v). \tag{3.4}$$

Let $\{\widetilde{\Phi}_l^i \mid i = 1, 2, \cdots, \widetilde{N}_l\}$ be the basis of $\widetilde{V}_l$. By the operator $\Xi_{l,\delta_{m(j)}}$, the basis of V_l consists of functions of the form

$$\Phi_l^i = \widetilde{\Phi}_l^i + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(\widetilde{\Phi}_l^i). \tag{3.5}$$

From the above definition, we can see that there exist two kinds of basis function of space V_l: (a) Φ_l^i and $\widetilde{\Phi}_l^i$ at all edges which are not in the interior of Γ are the same. Denote the set of this kind of basis functions by $\mathbf{\Phi}_0 = \{\Phi_l^i\}$; (b) Φ_l^i at all edges which are in the interior of each mortar edge $\gamma_{m(i)} \subset \Gamma$ are defined by (3.5). Denote the set of this kind of basis functions by $\mathbf{\Phi}_\Gamma = \{\Phi_l^i\}$.

Let $<\cdot, \cdot>$ be the Euclidean scalar product of the nodal basis in the finite element space V_l and denote the induced norm by $\|v\|_{0,d} = \sqrt{<v, v>}$. We define the operator $A_l: V_l \to V_l$ by

$$<A_l u, v> = a_l(u, v), \quad \forall u, v \in V_l,$$

which is represented in the basis by the stiffness matrix.

Following [6], we introduce a projection operator $P_l: V_{l-1} + V_l \to V_l$ defined by

$$a_l(P_l u, v) = a_l(u, v), \quad \forall v \in V_l.$$

From the definition, it is easily seen that

$$|P_l v|_l \leq |v|_{l-1}, \quad \forall v \in V_{l-1}. \tag{3.6}$$

We use the operator $\mathbf{C}_l^{m_l}: V_l \to V_l$ to denote m_l steps of iterations such as Gauss-Seidel, conjugate gradient method applied on level l. The cascadic multigrid method can be written as follows:

Cascadic multigrid algorithm

(1) Set $u_1^* = u_1$.

(2) Perform iterations, $l = 2, \cdots, L$.

$$u_l^0 = I_l u_{l-1}^*,\quad u_l^{m_l} = \mathbf{C}_l^{m_l} u_l^0.$$

(3) Set $u_l^* = u_l^{m_l}$.

Following [5], we call a cascadic multigrid optimal in the energy norm on the level L, if we obtain both the accuracy $|u_L - u_L^*|_L \approx |u - u_L|_L$, and the multigrid complexity amount of work $= O(n_L)$, $n_L = \dim(V_L)$. If the multigrid complexity is $O(n_L \log^\alpha n_L)$, where α is a fixed integer, the cascadic multigrid is nearly optimal

Shi and Xu[6] gave three hypothesis to guarantee the convergence of the cascadic multigrid method.

H1 For the intergrid transfer operator I_l, we assume that

(1) $\|v - I_l v\|_0 \lesssim h_l |v|_{l-1}$, $\forall v \in V_{l-1}$. (2) $\|u_l - I_l u_{l-1}\|_0 \lesssim h_l^2 \|f\|_0$.

where u_l is the mortar-type finite element solution of (3.1) on V_l,

H2 Assume that $u_l - \mathbf{C}_l^{m_l} u_l^0 = \mathbf{S}_l^{m_l}(u_l - u_l^0)$ with a linear mapping $\mathbf{S}_l^{m_l}: V_l \to V_l$ for the error propagation and for any $v \in V_l$

$$|\mathbf{S}_l^{m_l} v|_l \lesssim \frac{h_l^{-1}}{m_l^\gamma} \|v\|_0,\quad |\mathbf{S}_l^{m_l} v|_l \lesssim |v|_l,$$

where γ is a positive number depending on the given iteration.

H3 For the operator P_l, we assume that

$$\|v - P_l v\|_0 \lesssim h_l |v|_{l-1},\quad \forall v \in V_{l-1}.$$

Lemma 3.1 H1 holds for the mortar-type rotated Q_1 element space.

Proof H1-(1) has been obtained in [7]. Here we only need to prove H1-(2) is also valid.

From the triangle inequality, we get

$$\|u_l - I_l u_{l-1}\|_0 \lesssim \|u_l - \mathbf{\Pi}_l u_{l-1}\|_0 + \sum_{m=1}^{M} \|\Xi_{\delta_{m(j)}}(\mathbf{\Pi}_l u_{l-1})\|_0. \tag{3.7}$$

Using the similar argument in Lemma 3.3 in [10] and Theorem 2.1, the first term can be estimated

$$\|u_l - \mathbf{\Pi}_l u_{l-1}\|_0 \lesssim h_l^2 \|f\|_0. \tag{3.8}$$

By means of norm equivalence and the Schwarz inequality, we can derive

$$\begin{aligned} \|\Xi_{\delta_{m(j)}}(\mathbf{\Pi}_l u_{l-1})\|_0^2 &\lesssim h_l \|Q_{l,\delta_{m(j)}}(\mathbf{\Pi}_l u_{l-1}|_{\gamma_{m(i)}} - \mathbf{\Pi}_l u_{l-1}|_{\delta_{m(j)}})\|_{0,\gamma_m}^2 \lesssim \\ & h_l \Big(\|Q_{l,\delta_{m(j)}}(\mathbf{\Pi}_l u_{l-1}|_{\gamma_{m(i)}} - u_l|_{\gamma_{m(i)}})\|_{0,\delta_{m(j)}}^2 + \|Q_{l,\delta_{m(j)}}(u_l|_{\gamma_{m(i)}} - u_l|_{\delta_{m(j)}})\|_{0,\delta_{m(j)}}^2 + \\ & \|Q_{l,\delta_{m(j)}}(u_l|_{\delta_{m(j)}} - \mathbf{\Pi}_l u_{l-1}|_{\delta_{m(j)}})\|_{0,\delta_{m(j)}}^2 \Big). \end{aligned} \tag{3.9}$$

From the stability of the operator $Q_{l,\delta_{m(j)}}$, the trace theorem, the inverse inequality and the mortar condition, we have

$$\|\Xi_{\delta_{m(j)}}(\mathbf{\Pi}_l u_{l-1})\|_0 \le \|u_l - \mathbf{\Pi}_l u_{l-1}\|_{0,\Omega_i \cup \Omega_j}. \tag{3.10}$$

Combining (3.7)～(3.10), we get H1-(2).

Lemma 3.2 (1) Richardson, Jacobi and Gauss-Seidel iterations are smoothers in the sense of H2 with parameter $\gamma = 1/2$.

(2) H2 holds for the conjugate gradient iteration with $\gamma = 1$.

Proof (1) was shown in [11], now we prove (2) is valid. From Theorem 2.2 in [5], for the conjugate gradient method, we have

$$|\mathbf{S}_l^{m_l} v|_l \le \frac{\sqrt{\lambda_l}}{2m_l + 1} \|v\|_{0,d},\quad \text{and } |\mathbf{S}_l^{m_l} v|_l \le |v|_l,$$

where λ_l is the largest eigenvalue of A_l.

In order to complete the proof, we only need to prove the following results are valid

$$\lambda_l \le C, \quad \|v\|_{0,d} \lesssim h_l^{-1}\|v\|_0, \quad \forall v \in V_l. \tag{3.11}$$

Any $v \in V_l$ can be expressed by $v = v_0 + v_\Gamma = \sum_{\phi_l^i \in \Phi_0} \mu_i \phi_l^i + \sum_{\phi_l^i \in \Phi_\Gamma} \nu_i \phi_l^i$. Then, using the Schwarz inequality, we get

$$\langle A_l v, v \rangle = a_l(v, v) \le 2(a_l(v_0, v_0) + a_l(v_\Gamma, v_\Gamma)) \lesssim \sum_{\phi_l^i \in \Phi_0} \mu_i^2 a_l(\phi_l^i, \phi_l^i) + \sum_{\phi_l^i \in \Phi_\Gamma} \nu_i^2 a_l(\phi_l^i, \phi_l^i) \lesssim$$

$$\sum_{\phi_l^i \in \Phi_0} \mu_i^2 |\phi_l^i|_l^2 + \sum_{\phi_l^i \in \Phi_\Gamma} \nu_i^2 |\phi_l^i|_l^2. \tag{3.12}$$

Obviously, the supports of the basis functions in Φ_0 and Φ_Γ are $O(h_l^2)$. For each $\phi_l^i \in \Phi_0$, we can calculate directly $|\phi_l^i|_l = |\tilde{\phi}_l^i|_l \le C$. For each $\phi_l^i \in \Phi_\Gamma$, we have

$$|\phi_l^i|_l \le |\tilde{\phi}_l^i|_l + \sum_{m=1}^{M} |\Xi_{l,\delta_{m(j)}} \tilde{\phi}_l^i|_l. \tag{3.13}$$

Using the inverse inequality, the definition of $\Xi_{l,\delta_{m(j)}}$, the property of the operator $Q_{l,\delta_{m(j)}}$, the trace theorem and (1.3), we obtain

$$|\Xi_{l,\delta_{m(j)}} \tilde{\phi}_l^i|_l \lesssim h_{l,j}^{-1}\|\Xi_{l,\delta_{m(j)}} \tilde{\phi}_l^i\|_0 \lesssim h_{l,j}^{-1/2}\|Q_{l,\delta_{m(j)}} \tilde{\phi}_l^i\|_{0,\delta_{m(j)}} \lesssim h_{l,j}^{-1/2}\|\tilde{\phi}_l^i\|_{0,\gamma_{m(i)}} \lesssim h_l^{-1}\|\tilde{\phi}_l^i\|_{0,\Omega_i} \le C. \tag{3.14}$$

From (3.13) and (3.14), we have

$$|\phi_l^i|_l \le C, \quad \forall \phi_l^i \in \Phi_\Gamma. \tag{3.15}$$

Then

$$\langle A_l v, v \rangle \lesssim \sum_{\phi_l^i \in \Phi_0} \mu_i^2 + \sum_{\phi_l^i \in \Phi_\Gamma} \nu_i^2 = C\langle v, v \rangle, \text{ so } \lambda_l \le C.$$

By norm equivalence, we have

$$\|v\|_0^2 \gtrsim h_l^2 \sum_{E \in T_l} \sum_{e \in \partial E} \left(\frac{1}{|e|}\int_e v\,ds\right)^2 \gtrsim h_l^2 \|v\|_{0,d}^2,$$

so we complete the proof.

In the following, we use the duality argument to prove H3 holds for the finite element space. For this purpose, we consider the following auxiliary problem: for a given $v \in V_{l-1}$, find $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta\psi = v - P_l v & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Lemma 3.3 H3 holds for the mortar-type rotated Q_1 element space.

Proof Let ψ_l be the finite element approximation solution of the problem (3.16) in the discrete space V_l. From the definition of the operator P_l and Green's formula, we have

$$\|v - P_l v\|_0^2 = \sum_{E \in T_l} \int_E (v - P_l v)(-\Delta\psi)\,dx =$$

$$\sum_{E \in T_l} \int_E \nabla(v - P_l v)\nabla(\psi - \psi_l)\,dx - \sum_{E \in T_{l-1}} \int_{\partial E} \frac{\partial\psi}{\partial n} v\,ds + \sum_{E \in T_l} \int_{\partial E} \frac{\partial\psi}{\partial n} P_l v\,ds =$$

$$R_1 + R_2 + R_3. \tag{3.17}$$

For the first term at the right side of (3.17), we estimate directly as follows

$$|R_1| = |a_l(v - P_l v, \psi - \psi_l)| \le |v - P_l v|_l |\psi - \psi_l|_l \lesssim h_l |v - P_l v|_l \|\psi\|_2 \lesssim h_l |v|_{l-1}\|v - P_l v\|_0, \tag{3.18}$$

where (3.6) and the H^2-regularity assumption are used.

Following Lemma 4.2 in [3], we get

$$|R_2| \lesssim h_l\|v - P_l v\|_0 |v|_{l-1}, \text{ and } |R_3| \lesssim h_l\|v - P_l v\|_0 |P_l v|_l \lesssim h_l\|v - P_l v\|_0 |v|_{l-1}.$$

Then we obtain the result.

From Lemmas 3.1～3.3 and the framework given in [6], we have the following results

Theorem 3.1 The accuracy of the cascadic multigrid method can be estimated by

$$|u_L - u_L^*|_L \lesssim \sum_{l=2}^{L} \frac{h_l}{m_l^\gamma}\|f\|_0.$$

Let the number m_l of iteration steps on level l $(1 \leq l \leq L)$ be the smallest integer satisfying $m_l \geq \beta^{L-l} m_L$ for some fixed $\beta \geq 1$, where m_L is the number of iterations on the finest level L. We have

Theorem 3.2 The accuracy of the cascadic multigrid method is

$$\| u_L - u_L^* \|_L \lesssim \begin{cases} \dfrac{1}{1 - \dfrac{2}{\beta^{\gamma}}} \dfrac{h_L}{m_L^{\gamma}} \| f \|_0 & \beta > 2^{\frac{1}{\gamma}}, \\ L \cdot \dfrac{h_L}{m_L^{\gamma}} \| f \|_0 & \beta = 2^{\frac{1}{\gamma}}, \end{cases}$$

and the computational cost is proportional to

$$\sum_{l=2}^{L} m_l n_l \lesssim \begin{cases} \dfrac{1}{1 - \dfrac{\beta}{4}} m_L n_L & \beta < 4, \\ L \cdot m_L n_L & \beta = 4. \end{cases}$$

By Theorem 3.2, we know

Proposition 3.1 The cascadic multigrid method with the conjugate gradient method as the basic iteration (cascadic conjugate gradinet method) is optimal for $2 < \beta < 4$.

In the case $\gamma = \frac{1}{2}$, either the accuracy or the computational complexity has to deteriorate logarithmically. We choose to fix accuracy and obtain the following result as an immediate consequence of our results

Theorem 3.3 Let $\gamma = \frac{1}{2}$. If we choose the number of iterations on level L as $m_L = [m_* L^2]$, then we get $\| u_L - u_L^* \|_L \lesssim \frac{h_L}{m_*^{1/2}} \| f \|_0$, and as computational complexity $\sum_{l=2}^{L} m_l n_l \lesssim m_* n_L (1 + \log n_L)^3$.

From Theorem 3.3, we have

Proposition 3.2 The cascadic multigrid method with Richardson, Jacobi or Gauss-Seidel iterations as smoother is nearly optimal for $\beta = 4$.

4 Numerical Experiments

In this section we present some numerical results to illustrate the theory developed in the earlier sections. These examples deal with the Poisson equation on the unit square

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0, 1)^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f \in L^2(\Omega)$. For simplicity, we decompose Ω into two subdomains: $\Omega_1 = (0, 1) \times (0, \frac{1}{2})$ as mortar domain, and $\Omega_2 = (0, 1) \times (\frac{1}{2}, 1)$ as nonmortar domain. The meshsizes on the last level L are denoted by $h_{L,1}$ and $h_{L,2}$ respectively. Here we use Gauss-Seidel and conjugate gradient smoothing iterations and choose the exact solution of (4.1) as $u(x, y) = x(1 - x) y(1 - y)$, then $f(x, y) = 2x(1 - x) + 2y(1 - y)$.

The first test concerns the cascadic conjugate gradinet method (CCG). Let $\beta = 3$ and $m_L = 4$. From Tables 1, 2, we can see that if the mesh is refined one time, the energy error is decreasing by half independent of the coarse mesh.

Table 1 Error estimates for CCG with $h_{L,1}/h_{L,2}=2/3$

$h_{L,1}$	$h_{L,2}$	level(L)	$\vert u_L^* - u\vert_L$
0.020 833 3	0.031 25	3	0.005 254 31
		4	0.005 254 31
0.010 416 7	0.015 625	3	0.002 626 12
		4	0.002 626 63
0.005 208 33	0.007 812 5	3	0.001 311 93
		4	0.001 312 5

Table 2 Error estimates for CCG with $h_{L,1}/h_{L,2}=1/2$

$h_{L,1}$	$h_{L,2}$	level(L)	$\vert u_L^* - u\vert_L$
0.015 625	0.031 25	3	0.004 900 15
		4	0.004 900 15
0.007 812 5	0.015 625	3	0.002 446 76
		4	0.002 449 06
0.003 906 25	0.007 812 5	3	0.001 221 5
		4	0.001 223 33

The second test concerns the cascadic multigrid method with Gauss-Seidel iteration (CGS). We choose $\beta=4$ and $m_L=L^2$. In Tables 3, 4, if the mesh is refined one time, the energy error is also decreasing by half independent of the coarse mesh.

Table 3 Error estimates for CGS with $h_{L,1}/h_{L,2}=2/3$

$h_{L,1}$	$h_{L,2}$	level(L)	$\vert u_L^* - u\vert_L$
0.020 833 3	0.031 25	3	0.005 240 57
		4	0.005 240 58
0.010 416 7	0.015 625	3	0.002 620 07
		4	0.002 620 13
0.005 208 33	0.007 812 5	3	0.001 309 83
		4	0.001 309 95

Table 4 Error estimates for CGS with $h_{L,1}/h_{L,2}=1/2$

$h_{L,1}$	$h_{L,2}$	level(L)	$\vert u_L^* - u\vert_L$
0.015 625	0.031 25	3	0.004 877 65
		4	0.004 877 65
0.007 812 5	0.015 625	3	0.002 437 92
		4	0.002 438 18
0.003 906 25	0.007 812 5	3	0.001 218 64
		4	0.001 219 08

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