On the Siegel-Tatuzawa Theorem

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Abstract Siegel-Tatuzawa Theorem is an important result in proving the first one of the Gauss' conjectures on the imaginary quadratic number fields. After that, many people improved upon the result of Siegel-Tatuzawa Theorem. In this paper we use some Lemmas about the upper bound of $L(1,\chi)$ and some arithmetic theory of the biquadratic number field, get a lower bound of real primitive L-function at $s=1$. Let $0<\varepsilon<1/(6\log 10)$, and χ be a real primitive Dirichlet character modulo k which is greater than $e^{1/\varepsilon}$, then with at most one exception, the following expression holds:

$$L(1,\chi)>\min\left\{\frac{1}{7.702\log k},\ \frac{31.3\varepsilon}{k^{\varepsilon}}\right\}.$$

Key words L-function, real zeroes, quadratic number fields

CLC number O156.7 **Document code** A **Article ID** 1001-4616(2007)04-0032-04

关于 Siegel-Tatuzawa定理

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[摘要] Siegel-Tatuzawa定理是在研究 Gauss关于虚二次域类数的第一个猜想中产生的一个很重要的结论, Hoffstein等人对 Siegel-Tatuzawa定理的结果进行了改进,进一步得到了关于 $L(1,\chi)$下界的一些结论.本文在前人研究的基础上,利用 $L(1,\chi)$的上界以及双二次域的算术理论给出了对于实本原 Dirichlet特征 χ, $L(1,\chi)$较好的下界.

[关键词] L-函数,实零点,二次数域

0 Introduction

Let χ be a real primitive Dirichlet character modulo $k(>1)$. It is well known that if $L(s,\chi)$ has no zero in the interval $(1-c_1/\log k,\ 1)$, then $L(1,\chi)>c_2/\log k$, where c_1 and c_2 are positive constants and c_2 depends upon c_1 (see Lemma 1). If, however, $L(s,\chi)$ has a real zero close to 1, the only non-trivial lower bounds that are known for $L(1,\chi)$ are ineffective.

Siegel[1], for example, proved that for any $\varepsilon>0$,

$$L(1,\chi)>\frac{c(\varepsilon)}{k^{\varepsilon}},$$

where $c(\varepsilon)$ is an ineffective positive constant depending upon ε. After that Estermann[2], Chowla[3], Goldfeld[4] have given the simple proofs of Siegel's result, respectively.

Tatuzawa[5] proved that if $0<\varepsilon<1/11.2$ and $k>e^{1/\varepsilon}$, then with at most one exception,

Received date: 2007-04-15. **Revised date:** 2007-05-27.
Foundation item: Supported by the NNSF(10201013).
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$$L(1,\chi)>\frac{0.655\varepsilon}{k^{\varepsilon}}.$$

Hoffstein[6] proved that if $0<\varepsilon<1/(6\log 10)$ and $k>e^{1/\varepsilon}$, then with at most one exception,

$$L(1,\chi)>\min\{\frac{1}{7.735\log k},\ \frac{\varepsilon}{0.349k^{\varepsilon}}\}.$$

Lu and Ji[7] proved that if $0<\varepsilon<1/(6\log 10)$ and $k>e^{1/\varepsilon}$, then with at most one exception,

$$L(1,\chi)>\min\{\frac{1}{7.703\log k},\ \frac{18.236\varepsilon}{k^{\varepsilon}}\}.$$

Ji and Lu[8] proved that if $0<\varepsilon<1/(6\log 10)$ and $k>e^{1/\varepsilon}$, then with at most one exception,

$$L(1,\chi)>\min\{\frac{1}{7.7388\log k},\ \frac{32.260\varepsilon}{k^{\varepsilon}}\}.$$

Using a technique of Goldfeld[9], Lemma 2 in [8] and the upper bound estimate of $L(1,\chi)$ of Louboutin [10,11] and some arithmetic theory of the biquadratic number field, we prove the following

Theorem Let $0<\varepsilon<1/(6\log 10)$, and χ be a real primitive Dirichlet character modulo k which is greater than $e^{1/\varepsilon}$. Then with at most one exception, the following expression holds:

$$L(1,\chi)>\min\{\frac{1}{7.702\log k},\ \frac{31.3\varepsilon}{k^{\varepsilon}}\}.$$

1 Several Lemmas

Let $c_0=\frac{3}{2}+\sqrt{2}$

Lemma 1 (see [7, Lemma 1]) Let χ be a real primitive Dirichlet character modulo k which is greater than 10^6.

(1) If $L(s,\chi)\neq 0$ on the interval $(\beta, 1)$ and $1-\beta<1/(4c_0\log k)$, then $L(1,\chi)>1.5135(1-\beta)$.

(2) If $L(s,\chi)\neq 0$ on the interval $[3/4, 1)$, then $L(1,\chi)>1/(1.536\log k)$.

Lemma 2 (see [6, Lemma 2]) Let $K(\neq \mathbf{Q})$ be an algebraic number field and let d_K denote the absolute value of the discriminant of K. Then the Dedekind Zeta function $\zeta_K(s)$ of K has at most one real simple zero β with

$$1-\beta<\frac{1}{c_0\log d_K}.$$

Lemma 3 (see [8, Lemma 2]) Let K be an algebraic number field of degree $n>1$ and assume that for each $m\geqslant 1$ there exists at least one integral ideal of K of norm m^2 (e.g. K is a quadratic or a biquadratic bicyclic number field). Assume also that $1/2<\beta<1$ and $\zeta_K(\beta)\leqslant 0$. Then the residue at $s=1$ of the Dedekind Zeta function $\zeta_K(s)$ of K satisfies

$$h_K\geqslant(1-\beta)(x^{\beta-1}(\frac{\pi^2}{6}-\frac{n+2}{[\sqrt{x}]})-2\frac{d_K}{x^{3/2}}\frac{\zeta^n(3/2)(n+1)!}{(4n-3)\pi^n}),\quad(x\geqslant 1).$$

Lemma 4 (see [10] and [11]) (1) If χ is a primitive even Dirichlet character modulo $k>1$, then

$$|L(1,\chi)|\leqslant\frac{1}{2}(\log k+0.05).$$

(2) If χ is a primitive odd Dirichlet character modulo k, then

$$|L(1,\chi)|\leqslant\frac{1}{2}(\log k+1.44).$$

2 Proof of Theorem

Let $0<\varepsilon<1/(6\log 10)$ and let χ_1 be a real primitive Dirichlet character of least conductor $k_1>e^{1/\varepsilon}$ such that

$$L(1,\chi_1)\leqslant\frac{1}{7.702\log k_1},\tag{1}$$

if it exists. By Lemma 1, $L(s,\chi_1)$ has a real zero β_1 such that

$$1-\beta_1<\frac{1}{4c_0\log k_1}.\tag{2}$$

Let χ be another real primitive Dirichlet character modulo $k>e^{1/\varepsilon}$ different from χ_1. By our choice of k_1 we may assume $k\geqslant k_1$. Now we prove that

$$L(1,\chi)>\min\{\frac{1}{7.702\log k},\ \frac{31.3\varepsilon}{k^{\varepsilon}}\}$$

except χ_1. So it suffices to prove the above inequality when $k>k_1$. Let the Kronecker symbols of quadratic number fields $\mathbf{Q}(\sqrt{d})$, $\mathbf{Q}(\sqrt{d_1})$ be χ, χ_1 respectively, where d and d_1 are fundamental discriminants. Then $|d|=k$, $|d_1|=k_1$. Let $F=\mathbf{Q}(\sqrt{d},\sqrt{d_1})$, d_F denote the absolute value of the fundamental discriminant of F. Then it must exsit another quadratic number field $\mathbf{Q}(\sqrt{d_2})$ with the fundamental discriminant d_2 and $d_2\,|\,dd_1$, $|d_2|=|\frac{d_F}{dd_1}|$. If we let the Kronecker symbol of $\mathbf{Q}(\sqrt{d_2})$ be χ_2 with conductor $k_2=|d_2|$, then $\chi_2(n)=\chi(n)\chi_1(n)$. Then the Dedekind Zeta function of F is $\zeta_F(s)=\zeta(s)L(s,\chi)L(s,\chi_1)L(s,\chi_2)$, $s\in\mathbf{C}$ (see Chapter 4 of [12]). Let

$$\alpha=-3/2-\beta_1,\quad x=d_F^{A},\quad A>0.8,\tag{3}$$

and notice that $d_F\leqslant(kk_1)^2<k^4$, $kk_1>10^{12}$. Applying Lemma 3 with $n=4$, $x=d_F^{A}>d_F^{0.8}>10^{9.6}$ and $1-\beta_1<1/(4c_0\log k_1)<1/((6+4\sqrt{2})\log 10^6)<0.00621$, we obtain

$$1.60452(1-\beta_1)<k_Fx^{1-\beta_1}=L(1,\chi)L(1,\chi_1)L(1,\chi_2)x^{1-\beta_1}.\tag{4}$$

If $L(s,\chi)\neq 0$ in the range $1-1/(4c_0\log k)<s<1$, then by Lemma 1 we have

$$L(1,\chi)>\frac{1.5135}{4c_0\log k}>\frac{1}{7.702\log k},$$

and the result follows. If $L(s,\chi)=0$ for some β such that $1-1/(4c_0\log k)<\beta<1$ then both β and β_1 are zeroes of $\zeta_F(s)$. Since $\beta>1-1/(4c_0\log k)>1-1/(c_0\log d_F)$, we must have

$$1-\beta_1\geqslant\frac{1}{c_0\log d_F}\geqslant\frac{1}{2c_0\log(kk_1)}\tag{5}$$

by Lemma 2. Let $A=2/(\frac{3}{2}+\beta_1)$, then

$$2A(1-\beta_1)=\frac{4(1-\beta_1)}{2.5-(1-\beta_1)}<\frac{4}{10c_0-\frac{1}{\log k_1}}\cdot\frac{1}{\log k_1}<\frac{0.1376}{\log k_1}$$

by (2) and $k_1>10^6$, and we have

$$x^{1-\beta_1}=d_F^{A(1-\beta_1)}\leqslant(kk_1)^{2A(1-\beta_1)}<(kk_1)^{0.1376/\log k_1}.\tag{6}$$

From Lemma 4 we have

$$L(1,\chi_2)\leqslant\frac{1}{2}(\log k_2+1.44)\leqslant\frac{1}{2}(\log(kk_1)+1.44)<0.5261\log(kk_1)\tag{7}$$

for $kk_1>10^{12}$. Combining (1) and (4)~(7) we get

$$L(1,\chi)>\frac{4.03022}{\log k_1(1+\frac{\log k}{\log k_1})^2(kk_1)^{\frac{0.1376}{\log k_1}}}.\tag{8}$$

Let

$$\eta=\frac{\log k}{\log k_1},\quad \eta>1.$$

If $1<\eta\leqslant 7.5$, then by

$$\frac{4.030\,22\eta}{(\eta+1)^2 e^{0.137\,6(\eta+1)}}\cdot 7.702>1,$$

we get

$$L(1,\chi)>\frac{1}{7.702\log k}.$$

Hence we may assume $\eta>7.5$, in this case $2\log(\eta+1)<0.503\,6(\eta+1)$. Then

$$L(1,\chi)>\frac{2.122\,55}{(\log k_1)k^{0.641\,2/\log k_1}}. \tag{9}$$

For fixed $\delta>0$, $xe^{\delta/x}$ decreases until it reaches a minimum at $x=\delta$. Let $x=\log k_1$, $\delta=0.641\,2\log k$. Since $\log k>7.5\log k_1>(0.641\,2)^{-1}\log k_1$, $\log k_1>1/\varepsilon$. Hence

$$(\log k_1)k^{0.641\,2/\log k_1}<\frac{1}{\varepsilon}k^{0.641\,2\varepsilon}.$$

By (9), we have

$$L(1,\chi)>\frac{2.122\,55\varepsilon}{k^{0.641\,2\varepsilon}}=\frac{\varepsilon}{k^{\varepsilon}}\cdot 2.122\,55k^{0.358\,8\varepsilon}.$$

Since

$$2.122\,55k^{0.358\,8\varepsilon}>2.122\,55e^{0.358\,8\varepsilon(7.5\log k_1)}>2.122\,55e^{0.358\,8\cdot 7.5}>31.3,$$

we have

$$L(1,\chi)>\frac{31.3\varepsilon}{k^{\varepsilon}}.$$

This completes the proof of Theorem.

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