Simple Kirkman Frames With Index 2 and 3

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Abstract: A $(K, \ )$-frame is a GDD $(\mathscr{X}, \mathscr{G}, \mathscr{B})$ in which the collection of blocks $\mathscr{B}$ can be partitioned into holey parallel classes, each holey parallel class being a partition of $\mathscr{G} \backslash G_j$ for some G_j $\mathscr{G}$. A frame is called simple if all its blocks are distinct. Simple frames are powerful for the construction of simple Kirkman packing designs, which can be used in the construction of uniform designs in statistics. In this paper, we shall prove that the necessary conditions for simple Kirkman frames of type t^u with index 2 and 3 are also sufficient.

Key words: simple, frame design, group-divisible design

CLC number: O157.2　**Document code**: A　**Article ID**: 1001-4616(2008)01-0015-06

区组大小为 3的二重及三重单纯框架设计

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[摘要]　一个$(K, \ )$框架设计是一个区组集可分为若干个带洞平行类的 GDD $(\mathscr{X}, \mathscr{G}, \mathscr{B})$, 每一个带洞平行类为一个 $\mathscr{G} \backslash G_j$ 的划分, G_j $\mathscr{G}$. 若所有的区组是不同的, 则称框架设计是单纯的. 单纯的框架设计对构造单纯的可分解填充设计有很重要的作用, 后者可以用来构造统计学中的一致设计. 本文将证明 $(3, \ )$框架设计存在的必要条件也是充分的, 其中 $= 2, 3$.

[关键词]　单纯, 框架设计, 可分组设计

Let K be a set of positive integers. A group-divisible design $(K, \ )$-GDD is a triple $(\mathscr{X}, \mathscr{G}, \mathscr{B})$ which satisfies the following properties:

(1) $\mathscr{X}$ is a finite set of points;

(2) $\mathscr{G}$ is a partition of $\mathscr{X}$ into subsets called groups;

(3) $\mathscr{B}$ is a collection of subsets of $\mathscr{X}$ with sizes from K, called blocks, such that every pair of points from distinct groups occurs in exactly blocks, and

(4) No pair of points belonging to a group occurs in any block.

A $(K, \ )$-GDD $(\mathscr{X}, \mathscr{G}, \mathscr{B})$ is resolvable if the blocks of $\mathscr{B}$ can be partitioned into parallel classes, each parallel class being a partition of the point set $\mathscr{X}$. When $K = \{k\}$, we write $(K, \ )$-GDD as $(k, \ )$-GDD. Further, we denote $(K, 1)$-GDD as K-GDD and $(k, 1)$-GDD as k-GDD.

The type of the GDD $(\mathscr{X}, \mathscr{G}, \mathscr{B})$ is the multiset of sizes $|G|$ of the G $\mathscr{G}$ and we usually use the exponential notation for its description: type $1^i 2^j 3^k$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on. A transversal design TD(k, n) is a k-GDD of type n^k. It is well known that a TD(k, n) is equivalent to $k-2$ mutually orthogonal Latin squares of order n.

A $(K, \ )$-frame is a GDD $(\mathscr{X}, \mathscr{G}, \mathscr{B})$ in which the collection of blocks $\mathscr{B}$ can be partitioned into holey parallel classes, each holey parallel class being a partition of $\mathscr{X} \backslash G_j$ for some G_j $\mathscr{G}$. The groups in a $(K, \ )$-

Received date: 2007-08-28.

Foundation item: Supported by the NNSF(10501023, 60673070) and NSF of Jiangsu Province (BK2006217).

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frame are often referred to as holes. A uniform frame is a frame in which all groups are of the same size. A $(3,\ )$-frame is also called a Kirkman frame with index . In a $(3,\ )$-frame, it is not difficult to prove that to each group G_j there are exactly $|G_j|/2$ holey parallel classes that partition $\mathscr{X}\backslash G_j$.

A design is called simple if all its blocks are distinct. It is easy to see that the following conditions are necessary for the existence of a simple $(3,\ )$-frame of type t^u.

$$u\quad 4,$$
$$t(u-1)\quad 0(\mathrm{mod}\ 3),$$
$$t\quad 0(\mathrm{mod}\ 2)\ \text{and}$$
$$t(u-2)\quad .$$

Simple frames have been instrumental in the construction of other types of designs. For example, simple Kirkman frames can be used to construct simple Kirkman packing designs. Fang et al[1-2] have shown that simple Kirkman packing designs can be used in the construction of uniform designs in statistics.

It is obvious that a simple $(3,\ 1)$-frame is equivalent to a $(3,\ 1)$-frame. Stinson[3] has proved that there exists a $(3,\ 1)$-frame of type t^u if and only if t is even, $u\quad 4$, and $t(u-1)\quad 0(\mathrm{mod}\ 3)$. When $\quad 2$, not much results have been known. In [4], Shen has proved that there exits a simple $(3,\ 2)$-frame of type 1^u for any integer $u\quad 4$ and $u\quad 1(\mathrm{mod}\ 3)$.

Lemma 1 There exists a simple $(3,\ 2)$-frame of type 1^u for each $u\quad 1(\mathrm{mod}\ 3)$ and $u\quad 4$.

In this paper, we shall prove that the necessary conditions for simple Kirkman frames of type t^u with index 2 and 3 are also sufficient.

Theorem 1 There exists a simple $(3,\ 2)$-frame of type t^u if and only if $u\quad 4$ and $t(u-1)\quad 0(\mathrm{mod}\ 3)$.

Theorem 2 There exists a simple $(3,\ 3)$-frame of type t^u if and only if $u\quad 4$, t is even and $t(u-1)\quad 0(\mathrm{mod}\ 3)$.

1 Recursive Constructions

Now we state three basic constructions for simple frames. Similar proofs of these constructions can be found in [3] or [5].

Construction 1 (GDD construction): Let $(\mathscr{X}\ \mathscr{G}\ \mathscr{A})$ be a K-GDD, and let $w: \mathscr{X}\quad Z^+\quad \{0\}$ be a weighting function on $\mathscr{X}$. Suppose that for each block $A\quad \mathscr{A}$, there exists a simple $(k,\ )$-frame of type $\{w(x): x\quad A\}$. Then there exists a simple $(k,\ )$-frame of type $\{_{x\ G} w(x): G\quad \mathscr{G}\}$.

Construction 2 (Inflation by TDs): Suppose there is a simple $(k,\ )$-frame of type t^u, and suppose there is a resolvable TD$(k,\ n)$. Then there is a simple $(k,\ )$-frame of type $(nt)^u$.

Construction 3 (Filling in holes): Suppose there is a simple $(k,\ )$-frame with groups of sizes from $T=\{t_1,\ \ ,\ t_n\}$ and let $\quad 0$. For $1\quad i\quad n$, suppose there is a simple $(k,\ )$-frame with groups of sizes from $T_i\quad \{\ \}$, where $_{t\ T_i} t = t_i$. Then there is a simple $(k,\ )$-frame with groups of sizes from $(\ _{i=1}^{n} T_i)\quad \{\ \}$.

In order to use the above constructions, we need the notion of PBD. A pairwise balanced design $(v,\ K)$-PBD is a pair $(\mathscr{X}\ \mathscr{B})$, in which $\mathscr{X}$ is a set of v points and $\mathscr{B}$ is a set of blocks with sizes from K, such that every unordered pair of points is contained in a unique block. A $(v,\ K)$-PBD is a K-GDD of type 1^v indeed. From [6], we have the following known results.

Lemma 2 (1) There exists a $(v,\ \{4,\ 7\})$-PBD for every $v\quad 1(\mathrm{mod}\ 3)$ and $v \notin \{10,\ 19\}$. (2) There exists a $(v,\ \{4,\ 7\})$-PBD which contains a unique block of size 7 for all these values $v\quad 7$ or $10(\mathrm{mod}\ 12)$, $v\quad 7$ and $v\quad \{10,\ 19\}$.

Lemma 3 There exists a $\{4,\ 7\}$-GDD of type 3^u for each $u\quad 2$ or $3(\mathrm{mod}\ 4)$ and $u\quad \{3,\ 6\}$.

Proof By Lemma 2, there exists a $(3u+1,\ \{4,\ 7\})$-PBD which contains a unique block of size 7 for every $u\quad 2$ or $3(\mathrm{mod}\ 4)$, $u\quad 2$ and $u\quad \{3,\ 6\}$. Let x be a point which doesn t occur in the block of size 7. It is easy to see that there are exactly u blocks containing x. Deleting this point, we get the required $\{4, 7\}$-GDD of type 3^u.

To obtain our main results, we also need the following known results, which can be found in [7] and [8].

Lemma 4 (1) There exists a 4-GDD of type 2^u for each u $1 (\mathrm{mod}\ 3)$ and $u > 4$. (2) There exists a 4-GDD of type 3^u for each u $0, 1 (\mathrm{mod}\ 4)$ and u 4. (3) There exists a 4-GDD of type 6^u for any integer u 5.

Lemma 5 A resolvable TD$(3, n)$ exists for all positive integers except for $n = 2, 6$.

2 Uniform Simple (3, 2)-Frame

In this section, we shall prove Theorem 1. We need the following two direct constructions. For direct constructions, instead of listing all the blocks and the holey parallel classes of the desired simple frames, we only list the blocks of the initial holey parallel classes; the other holey parallel classes can be generated under some additive group G.

Lemma 6 There exists a simple $(3, 2)$-frame of type 2^4.

Proof Let the point set $V = Z_8$, and the groups be $G_j = \{0+j, 4+j\}$, j Z_4. The required 8 holey parallel classes will be generated from the following two initial holey parallel classes by $+2 \bmod 8$. The blocks of the two initial holey parallel classes are listed below.

$$\begin{array}{lll} 0\ 1\ 2 & 4\ 5\ 6 & (3, 7) \\ 1\ 3\ 6 & 2\ 5\ 7 & (0, 4) \end{array}$$

Lemma 7 There exists a simple $(3, 2)$-frame of type 3^6.

Proof Let the point set $V = Z_{18}$, and the groups be $G_j = \{0+j, 6+j, 12+j\}$, j Z_6. The required 18 holey parallel classes will be generated from the following two initial holey parallel classes by $+2 \bmod 18$. The blocks of the two initial holey parallel classes are listed below.

$$\begin{array}{llllll} 1\ 2\ 3 & 4\ 5\ 7 & 8\ 10\ 15 & 9\ 13\ 17 & 11\ 14\ 16 & (0, 6, 12) \\ 0\ 3\ 10 & 2\ 11\ 16 & 4\ 8\ 15 & 5\ 6\ 14 & 9\ 12\ 17 & (1, 7, 13) \end{array}$$

Lemma 8 There exists a simple $(3, 2)$-frame of type 2^u for each u $1 (\mathrm{mod}\ 3)$ and u 4.

Proof By Lemma 4, we have a 4-GDD of type 2^u for each u $1 (\mathrm{mod}\ 3)$ and $u > 4$. Applying Construction 1 with $w = 1$, we get a simple $(3, 2)$-frame of type 2^u for each u $1 (\mathrm{mod}\ 3)$ and $u > 4$, the input simple $(3, 2)$-frame of type 1^4 comes from Lemma 1. From Lemma 6, there exists a simple $(3, 2)$-frame of type 2^4. The proof is completed.

Lemma 9 There exists a simple $(3, 2)$-frame of type 3^u for each u 4.

Proof We distinguish 2 cases.

Case 1. u 0 or 1 $(\mathrm{mod}\ 4)$.

By Lemma 4, we have a 4-GDD of type 3^u for each u 4. Applying Construction 1, we get a simple $(3, 2)$-frame of type 3^u for each u 4.

Case 2. u 2 or 3 $(\mathrm{mod}\ 4)$.

By Lemma 3, we have a $\{4, 7\}$-GDD(3^u) for each u 2 or 3$(\mathrm{mod}\ 4)$ and $u > 6$. Applying Construction 1, we get a simple $(3, 2)$-frame of type 3^u for each u $2, 3 (\mathrm{mod}\ 4)$ and $u > 6$. From Lemma 7, there exists a simple $(3, 2)$-frame of type 3^6. Here, the input simple $(3, 2)$-frames of type 1^4 and 1^7 exist by Lemma 1. This completes the proof.

Lemma 10 There exists a simple $(3, 2)$-frame of type 6^u for each u 4.

Proof By Lemma 4, we have a 4-GDD of type 6^u for each u 5. Applying Construction 1, we get a simple $(3, 2)$-frame of type 6^u for each u 5. This leaves the only case $u = 4$ to be considered. By Lemma 4, we have a 4-GDD of type 3^4. Applying Construction 1 with $w = 2$, we get a simple $(3, 2)$-frame of type 6^4, the input simple $(3, 2)$-frame of type 2^4 comes from Lemma 6. The conclusion then follows.

Now we are in the position to prove Theorem 1.

Theorem 3 There exists a simple $(3, 2)$-frame of type t^u if and only if u 4, and $t(u-1)$ 0 (mod 3).

Proof We distinguish 2 cases.

Case 1. t 0 (mod 3), u 1 (mod 3) and u 4.

By Lemma 1, there exists a simple $(3, 2)$-frame of type 1^u for any u 1 (mod 3) and u 4. Applying Construction 2, we obtain a simple $(3, 2)$-frame of type t^u, t 2. The input resolvable TD$(3, t)$ exists by Lemma 5. By Lemma 8, there exists a simple $(3, 2)$-frame of type 2^u for any u 1 (mod 3) and u 4.

Case 2. t 0 (mod 3), u 4.

Let $t=3k$. If t 6, 18, then k 2, 6. By Lemma 9, there exists a simple $(3, 2)$-frame of type 3^u for any u 4. Applying Construction 2, we obtain a simple $(3, 2)$-frame of type $(3k)^u$, k 2, 6. The input resolvable TD$(3, k)$ exists by Lemma 5. By Lemma 10, there exists a simple $(3, 2)$-frame of type 6^u for any u 4. This leaves the only case $t=18$ to be considered. Start with a simple $(3, 2)$-frame of type 6^u. Applying Construction 2 again, we obtain a simple $(3, 2)$-frame of type $(18)^u$, the input resolvable TD$(3, 3)$ exists by Lemma 5. This completes the proof.

3 Uniform Simple (3, 3)-Frame

In this section, we shall prove Theorem 2. We first give some direct constructions.

Lemma 11 There exists a simple $(3, 3)$-frame of type 2^u for each u $\{4, 7, 10\}$.

Proof Let the point set $V=Z_{2u}$ and the groups be $G_j=\{0+j, u+j\}$, j Z_u, u $\{4, 7, 10\}$. The required $3u$ holey parallel classes will be generated from the following initial holey parallel classes by +1 mod 4, +2 mod 14 and +4 mod 20 respectively. The blocks of the initial holey parallel classes for each u are listed below.

$i=4$:	1 2 7	3 5 6	(0, 4)				
	1 2 3	5 6 7	(0, 4)				
	1 3 6	2 5 7	(0, 4)				
$i=7$:	1 2 3	4 5 6	8 10 13	9 11 12	(0, 7)		
	1 3 11	2 6 12	4 10 13	5 8 9	(0, 7)		
	1 4 9	2 5 11	3 8 13	6 10 12	(0, 7)		
$i=10$:	1 2 3	4 5 6	7 8 9	11 12 14	13 15 17	16 18 19	(0, 10)
	1 4 7	2 5 8	3 12 16	6 13 17	9 14 18	11 15 19	(0, 10)
	1 12 17	2 7 15	3 9 14	4 13 18	5 11 16	6 8 19	(0, 10)
	0 3 4	2 5 6	7 9 12	8 13 15	10 16 17	14 18 19	(1, 11)
	0 4 12	2 5 13	3 9 17	6 15 18	7 10 19	8 14 16	(1, 11)
	0 7 13	2 9 16	3 10 18	4 15 19	5 12 17	6 8 14	(1, 11)

Lemma 12 There exists a simple $(3, 3)$-frame of type 4^4.

Proof Let the point set $V=Z_{16}$ and the groups be $G_j=\{0+j, 4+j, 8+j, 12+j\}$, j Z_4. The required 24 holey parallel classes will be generated from the following three initial holey parallel classes by +1 mod 16. The blocks of the three initial holey parallel classes P_1, P_2, P_3 are listed below. Here, P_i ($i=2, 3$) will generate 4 holey parallel classes by +1 mod 16.

P_1:	1 2 7	3 5 10	6 9 15	11 13 14	(0, 4, 8, 12)
P_2:	2 3 9	6 7 13	1 10 11	5 14 15	(0, 4, 8, 12)
P_3:	1 3 6	5 7 10	9 11 14	2 13 15	(0, 4, 8, 12)

Lemma 13 There exists a simple $(3, 3)$-frame of type 6^6.

Proof Let the point set $V=Z_{36}$ and the groups be $G_j=\{0+j, 6+j, 12+j, 18+j, 24+j, 30+j\}$, j Z_6. The required 54 holey parallel classes will be generated from the following six initial holey parallel classes by

+ 4 mod 36. The blocks of the six initial holey parallel classes are listed below.

1 5 10	2 7 16	3 14 17	4 21 26	8 23 25	9 32 35
11 27 28	13 20 34	15 19 29	22 31 33		(0, 6, 12, 18, 24, 30)
1 2 3	4 5 9	7 11 14	8 17 27	10 13 26	15 28 29
16 23 31	19 32 34	20 25 33	21 22 35		(0, 6, 12, 18, 24, 30)
1 9 22	2 13 28	3 11 16	4 32 33	5 7 14	8 27 29
10 25 26	15 19 35	17 20 34	21 23 31		(0, 6, 12, 18, 24, 30)
0 4 11	2 6 15	3 8 12	5 21 28	9 20 29	10 18 27
14 24 33	16 23 32	17 26 34	22 30 35		(1, 7, 13, 19, 25, 31)
0 3 4	2 5 9	6 8 11	10 12 20	14 15 18	16 26 27
17 28 33	21 24 35	22 29 32	23 30 34		(1, 7, 13, 19, 25, 31)
0 2 28	3 8 24	4 21 29	5 18 20	6 23 26	9 14 35
10 17 27	11 22 33	12 32 34	15 16 30		(1, 7, 13, 19, 25, 31)

Lemma 14 There exists a simple (3, 3)-frame of type 4^u for each u $\{7, 10\}$.

Proof By Lemma 4, we have a 4-GDD of type 2^u for each u $\{7, 10\}$. Applying Construction 1 with $w = 2$, we get a simple (3, 3)-frame of type 2^u, the input simple (3, 3)-frame of type 2^4 comes from Lemma 11. The proof is completed.

Similarly, starting from a 4-GDD of type 6^6 which exists by Lemma 4 and applying Construction 1 with $w = 2$, we get a simple (3, 3)-frame of type 12^6.

Lemma 15 There exists a simple (3, 3)-frame of type 12^6.

Lemma 16 There exists a simple (3, 3)-frame of type 6^u for each u 4.

Proof We distinguish 2 cases.

Case 1. u 0 or 1 (mod 4).

By Lemma 4, we have a 4-GDD of type 3^u for each u 4. Applying Construction 1 with $w = 2$, we get a simple (3, 3)-frame of type 6^u for each u 4.

Case 2. u 2 or 3 (mod 4).

By Lemma 3, we have a {4, 7}-GDD(3^u) for each u 2 or 3 (mod 4) and $u > 6$. Applying Construction 1 with $w = 2$, we get a simple (3, 3)-frame of type 6^u for each u 2, 3 (mod 4) and $u > 6$. Here, the input simple (3, 3)-frames of type 2^4 and 2^7 exist by Lemma 11. From Lemma 13, there exists a simple (3, 3)-frame of type 6^6. This completes the proof.

Similarly, we can apply Construction 1 with $w = 4$ to get a simple (3, 3)-frame of type 12^u for each u 4, where the input simple (3, 3)-frames of type 4^4 and 4^7 come from Lemma 12 and Lemma 14, and a simple (3, 3)-frame of type 12^6 exists by Lemma 15.

Lemma 17 There exists a simple (3, 3)-frame of type 12^u for each u 4.

Lemma 18 There exists a simple (3, 3)-frame of type 2^u for each u 1 (mod 3) and u 4.

Proof Let $u = 3k + 1$ (k 1). By Lemma 16, there exists a simple (3, 3)-frame of type 6^k for each k 4. Applying Construction 3 with $= 2$, we get a simple (3, 3)-frame of type 2^{3k+1} for each k 4. By Lemma 11, there exists simple (3, 3)-frames of type 2^4, 2^7 and 2^{10}. The conclusion then follows.

Similarly, we can apply Construction 3 with $= 4$ to get a simple (3, 3)-frame of type 4^u for each u 1 (mod 3) and u 4, where the input simple (3, 3)-frames come from Lemma 12, Lemma 14 and Lemma 17.

Lemma 19 There exists a simple (3, 3)-frame of type 4^u for each u 1 (mod 3) and u 4.

Now we are in the position to prove Theorem 2.

Theorem 4 There exists a simple (3, 3)-frame of type t^u if and only if u 4, t is even and $t(u-1)$ 0 (mod 3).

Proof We have proved that the necessary conditions are also sufficient for t $\{2, 4, 6, 12\}$ by Lemma 18, Lemma 19, Lemma 16 and Lemma 17. Let t $\{2, 4, 6, 12\}$.

If t 0 (mod 6), then write $t = 6t_1$ or $12t_1$, where t_1 2, 6. By Lemma 5, there exists a resolvable TD(3, t_1). Start from a simple (3, 3)-frame of type 6^u or 12^u and apply Construction 2 with $(k, n) = (3, t_1)$. The result is a simple (3, 3)-frame of type $(6t_1)^u$ or $(12t_1)^u$.

If t 2 or 4 (mod 6), then write $t = 2t_1$, where t_1 2, 6. Similarly, we start from a simple (3, 3)-frame of type 2^u and apply Construction 2 to obtain a simple (3, 3)-frame of type t^u. This completes the proof.

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