Positive Solutions of Elliptic and Parabolic Problems With Nonlocal Lower Order Terms

Shang Xudong, Zhang Jihui

(School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210097, China)

Abstract The existence of positive solution about nonlocal boundary problems with lower order nonlocal terms

$$\begin{cases} -a\left(\int_\Omega |u|^q dx\right)\Delta u + b(l(u))u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and its parabolic counterpart were considered where Ω was a bounded smooth domain of $\mathbf{R}^N$ $(N > 2)$, a and b were given function. By using Galerkin method, the existence of positive solution for the nonlocal elliptic problems was obtained, moreover, the existence of positive solution for the evolution case was proved.

Key words: nonlocal problem, boundary value problem, positive solution, Galerkin method

CLC number O175.26 **Document code** A **Article ID** 1001-4616(2008)02-0008-05

具有低阶项的非局部椭圆及抛物问题的正解

尚旭东, 张吉慧

(南京师范大学数学与计算机科学学院, 江苏 南京 210097)

[摘要] 考虑了非局部边值问题

$$\begin{cases} -a\left(\int_\Omega |u|^q dx\right)\Delta u + b(l(u))u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

及其相应的非局部抛物问题的正解存在性. 其中 Ω 是 $\mathbf{R}^N$ 中的有界光滑区域, a 和 b 是给定的函数. 利用 Galerkin 方法, 首先获得了具有低阶项的非局部椭圆问题正解的存在性, 进一步证明了抛物问题正解的存在性.

[关键词] 非局部问题, 边值问题, 正解, Galerkin方法

In this paper we concerned with the elliptic problems with nonlocal lower order terms

$$\begin{cases} -a\left(\int_\Omega |u|^q dx\right)\Delta u + b(l(u))u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and its parabolic counterpart

$$\begin{cases} u_t - a\left(\int_\Omega |u|^q dx\right)\Delta u + b(l(u))u = f(x, u), & \text{in } Q_T = \Omega \times (0, T), \\ u(x, t) = 0, & \text{on } \Gamma = \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \geqslant 0, & \text{in } \Omega, \end{cases} \tag{2}$$

where $1 \leqslant q < 2N/(N-2)$, $a: \mathbf{R} \to (0, \infty)$, $b: \mathbf{R} \to (0, \infty)$ and $f: \Omega \times \mathbf{R} \to (0, \infty)$ are given functions who-

Received date: 2007-11-28.

Foundation item: Supported by Foundation of Major Project of Science and Technology of Chinese Education Ministry (205101SBJBC51), SRFDP of Higher Education (20051011TSJB157).

Corresponding author: Zhang Jihui, professor, majored in virational method and PDE. E-mail: jihuiz@jlonline.com

ses properties will be timely introduced, $l: L^2(\Omega) \to \mathbf{R}$ is a continuous linear form, Ω is a bounded smooth domain of $\mathbf{R}^N$ $(N > 2)$. These problems arise in various physical situation and biological models. For instance when we study question related with a culture of bacterias, u could describe the population of these bacterias and so $u > 0$, in Ω. Where the diffusion coefficient a is supposed to depend on the entire population in the domain rather than on the local density and the $b(l(u))u$ is the density of death or extinction of the population at stake. It's relatively normal to assume this death to the proportional to the density of population by a factor b. Now this factor could have $b = b(\int_\Omega u(x)\,dx)$, b depends on the entire population. In the recent years, several papers have been devoted to the study of nonlocal elliptic and parabolic problems[1-7]. But in this paper, we consider the elliptic and parabolic problems with nonlocal lower order terms, using Galerkin method obtained the positive solution. Throughout this paper we will suppose that

(C_1) $a(s)$ is a continuous function and $0 < m_0 \leqslant a(s) \leqslant M$, $\forall\ s \in \mathbf{R}$;

(C_2) $b(s)$ is a continuous function and $0 < \alpha \leqslant b(s) \leqslant \beta$, $\forall\ s \in \mathbf{R}$;

(C_3) $l: L^2(\Omega) \to \mathbf{R}$ is a continuous linear form, there is a $g(x) \in L^2(\Omega)$ such that $l(u) = l_g(u) = \int_\Omega g(x)u(x)\,dx$ for all $u \in L^2(\Omega)$.

1 The Stationary Problem

In this section we are going to considered problem (1) by using a Galerkin method. We can see in [8, 9] about Galerkin method in nonlocal problem. Concerning to problem (1) we will suppose that

(H_1) $f: \Omega \times \mathbf{R} \to (0, \infty)$ is a Carathodory function and satisfying $f(x, u) \leqslant h(x)u(x) + g(x)$, where $h(x), g(x) \in C(\Omega)$ and $h, g > 0$ in Ω, $a = \max h(x)$, $b = \max g(x)$.

Under these assumptions, we say a function $u \in H_0^1(\Omega)$ is called weak solution of problem (1) if for $\forall\ \phi \in H_0^1(\Omega)$,

$$a\left(\int_\Omega |u|^q\,dx\right)\int_\Omega \nabla u \nabla \phi\,dx + b(l(u))\int_\Omega u\phi\,dx - \int_\Omega f(x, u)\phi\,dx = 0. \tag{3}$$

Theorem 1 Let assume that condition (C_1), (C_2) and (C_3) hold. Let assume in addition that (H_1) and $m_0 < \frac{a-\alpha}{\lambda_1}$, $a > \alpha$, where λ_1 is the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Then problem (1) has at least one positive solution.

Proof Let $\{v_k\}_{k=1}^{\infty}$ be a complete orthonormal system for $H_0^1(\Omega)$ with norm $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx$. For each fixed $n \in \mathbf{N}$, consider the finite-dimensional Hilbert space $V_n = \mathrm{span}\{v_1, \cdots, v_n\}$. Then V_n is isometric to $\mathbf{R}^n$ for each $v \in V_n$, is uniquely associate to $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$ by the relation $v = \sum_{k=1}^{n} \xi_k v_k$. We get $\|u\| = |\xi|_{\mathbf{R}^n}$. We search for solution $u_n \in V_n$ of the approximate problem

$$a\left(\int_\Omega |u_n|^q\,dx\right)\int_\Omega \nabla u_n \nabla v_k\,dx + b(l(u_n))\int_\Omega u_n v_k\,dx - \int_\Omega f(x, u_n)v_k\,dx = 0. \tag{4}$$

Consider the function $F_n: \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$(F_n u)_k = a\left(\int_\Omega |u|^q\,dx\right)\int_\Omega \nabla u \nabla v_k\,dx + b(l(u))\int_\Omega u v_k\,dx - \int_\Omega f(x, u)v_k\,dx, \tag{5}$$

$\forall\ u \in V_n$. By (H_1), (C_1) and (C_2), we can easily get F_n is continuous and

$$\langle F_n u, u\rangle = a\left(\int_\Omega |u|^q\,dx\right)\int_\Omega |\nabla u|^2\,dx + b(l(u))\int_\Omega u^2\,dx - \int_\Omega f(x, u)u\,dx,$$

Using (H_1), Poincar and Hlder inequalities, we get

$$\langle F_n u, u\rangle \geqslant \left(m_0 - \frac{a-\alpha}{\lambda_1}\right)\|u\|^2 - b\,|\Omega|^{\frac{1}{2}}\lambda_1^{\frac{-1}{2}}\|u\| > 0,$$

if $\|u\| = R$, for R large enough, independently of n, where $|\Omega|$ is the lebesgue measure of the set Ω. Thus, by using Brouwer fixed point theorem in [10], then (5) has a solution $u_n \in V_n$ and $\|u_n\| \leqslant R$, $\forall\ n \in \mathbf{N}$. From the completeness of v_n, which implies that

$$a\left(\int_\Omega |u_n|^q dx\right)\int_\Omega \nabla u_n \nabla v dx + b(l(u_n))\int_\Omega u_n v dx - \int_\Omega f(x, u_n)v dx = 0, \tag{6}$$

$\forall v \in H_0^1(\Omega)$, let us prove that the sequence $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence which converges to a solution of (1). In fact, since $\{u_n\}$ is a bounded, there exists a subsequence still denoted by (u_n), such that

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega), \quad u_n \to u \text{ in } L^2(\Omega),$$
$$u_n \to u \text{ in } L^p(\Omega),\ 1 \leqslant p < 2N/(N-2), \quad u_n(x) \to u(x) \text{ a. e. in } \Omega,$$
$$\|u_n\|_p^p \to \|u\|_p^p,$$

and by (C_3), we has that $l(u_n) \to l(u)$. Hence, using above and passing to the limit in (6), it shows that u is a weak solution of problem (1). Next we prove $u > 0$. Set $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. Then $u = u^+ + u^-$. Because u is a weak solution of (1) and let $\phi = u^-$ in (3), we can get

$$a\left(\int_\Omega |u|^q dx\right)\int_\Omega |\nabla u^-|^2 dx + b(l(u))\int_\Omega |u^-|^2 dx = \int_\Omega f(x, u)u^- dx.$$

From the assumption we obtain

$$\int_\Omega f(x, u)u^- dx \geqslant 0.$$

But $f(x, u) > 0$ and $u^- \leqslant 0$, hence, $\int_\Omega f(x, u)u^- dx = 0$. We have

$$u^- = 0 \text{ i. e. } u \geqslant 0.$$

So u is a positive solution.

2 The Parabolic Problem

In this section, we consider the parabolic problem (2). First we give a definition of weak solution bellow. In which assume satisfies the addition condition

(H_2) $f: \Omega \times \mathbf{R} \to (0, \infty)$ is a Carathodory function and satisfying $|f(x, u)| \leqslant c(1+|u|)$.

Definition 1 We say a function $u \in L^2(0, T; H_0^1(\Omega))$ with $u' \in L^2(0, T; H^{-1}(\Omega))$ is a weak solution of (2), provided

(i) $$\int_\Omega u' v dx + a\left(\int_\Omega |u|^q dx\right)\int_\Omega \nabla u \nabla v dx + b(l(u))\int_\Omega uv dx = \int_\Omega f(x, u)v dx, \tag{7}$$

for each $v \in H_0^1(\Omega)$ and a. e. time $0 \leqslant t \leqslant T$.

(ii) $u(0) = u_0(x)$.

Lemma 1 If $u \in L^2(0, T; H_0^1(\Omega))$ with $u' \in L^2(0, T; H^{-1}(\Omega))$ is a nontrivial solution of problem (2), then $u(x, t) > 0$, $x \in \Omega$, $t \geqslant 0$.

Proof Multiplying u^- by Eq(7), and integrating over Q_T, we can see

$$\int_\Omega\int_0^T \frac{\partial u}{\partial t}u^- dtdx + a\left(\int_\Omega |u|^q dx\right)\int_\Omega\int_0^T \nabla u \nabla u^- dtdx + b(l(u))\int_\Omega\int_0^T uu^- dtdx =$$
$$\int_\Omega\int_0^T f(x, u)u^- dtdx.$$

Considering the above terms respectively, because $u_0(x) > 0$,

$$\int_\Omega\int_0^T \frac{\partial u}{\partial t}u^- dtdx = \frac{1}{2}\int_\Omega (u^-(x, T))^2 dx > 0,$$

$$a\left(\int_\Omega |u|^q dx\right)\int_\Omega\int_0^T \nabla u \nabla u^- dtdx = a\left(\int_\Omega |u|^q dx\right)\|u^-\|^2_{H_0^1(0, T; L^2(\Omega))} > 0,$$

$$b(l(u))\int_\Omega\int_0^T uu^- dtdx = b(l(u))\|u^-\|^2_{L(0, T; L^2(\Omega))} > 0.$$

Thus

$$\int_\Omega\int_0^T f(x, u)u^- dtdx > 0.$$

But

$$\int_\Omega\int_0^T f(x, u)u^- \,\mathrm{d}t\mathrm{d}x \leqslant 0.$$

Hence, we get $u^- = 0$, it means $u > 0$.

Theorem 2 Let assume that condition (C_1), (C_2), (C_3) and (H_2) hold. Then problem (2) has at least one positive solution.

Proof We use Galerkin method. Let $\{w_k\}_{k=1}^{\infty} \subseteq H_0^1(\Omega)$ is an orthogonal basis of $H_0^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. Fix now a positive integer $m \in \mathbf{N}$, set

$$V_m = \mathrm{span}\{w_1, \cdots, w_m\}.$$

We note

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k(x), \tag{8}$$

and $u_{0m} \to u_0$ in $H_0^1(\Omega)$ to be the solution of approximate problem

$$\int_\Omega u'_m w_k \,\mathrm{d}x + a\left(\int_\Omega |u_m|^q \,\mathrm{d}x\right)\int_\Omega \nabla u_m \nabla w_k \,\mathrm{d}x + b(l(u_m))\int_\Omega u_m w_k \,\mathrm{d}x = \int_\Omega f(x, u_m) w_k \,\mathrm{d}x, \tag{9}$$

$$u_m(0) = u_{0m}, \quad u_m \in V_m. \tag{10}$$

According to stand existence theory for linear ordinary different equations, there (9) has a unique solution u_m. We now to show a subsequence of our solution (u_m) converges to a weak solution of problem (2). For this we will need some priori estimate. Multiply (9) by $d_m^k(t)$, sum for $k = 1, \cdots, m$. We can find

$$\int_\Omega u'_m u_m \,\mathrm{d}x + a\left(\int_\Omega |u_m|^q \,\mathrm{d}x\right)\int_\Omega \nabla u_m \nabla u_m \,\mathrm{d}x + b(l(u_m))\int_\Omega u_m^2 \,\mathrm{d}x = \int_\Omega f(x, u_m) u_m \,\mathrm{d}x,$$

$0 \leqslant t \leqslant T$. By using (H_2) and Cauchy's inequality with ϵ, take $\epsilon = 4\alpha$, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m\|_{L^2(\Omega)}^2\right) + 2m_0\|u_m\|_{H_0^1(\Omega)}^2 \leqslant 2c\|u_m\|_{L^2(\Omega)}^2 + 2c^2|\Omega|. \tag{11}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_m(t)\|_{L^2(\Omega)}^2\right) \leqslant 2c\|u_m\|_{L^2(\Omega)}^2 + 2c^2|\Omega|.$$

By using Gronwall's inequality we have

$$\|u_m(t)\|_{L^2(\Omega)}^2 \leqslant C_1\left(\|u_0\|_{L^2(\Omega)}^2 + 2c^2|\Omega|T\right). \tag{12}$$

Integrate (11) from 0 to T and using (12) we obtain

$$\|u_m\|_{L^2(\Omega)}^2 + 2m_0\|u_m\|_{L^2(0,T;H_0^1(\Omega))}^2 \leqslant C_2\left(\|u_0\|_{L^2(\Omega)}^2 + 2c^2|\Omega|T\right).$$

Hence

$$\|u_m\|_{L^2(0,T;H_0^1(\Omega))}^2 = \int_0^T \|u_m(t)\|_{H_0^1(\Omega)}^2 \,\mathrm{d}t \leqslant C_3\left(\|u_0\|_{L^2(\Omega)}^2 + 2c^2|\Omega|T\right). \tag{13}$$

Fix any $v \in H_0^1(\Omega)$ with $\|v\|_H \leqslant 1$ and $v = v^1 + v^2$, where $v^1 \in V^m$ and $(v^2, w_k) = 0\ (k = 1, \cdots, m)$, by (9) we get that

$$\int_\Omega \frac{\mathrm{d}u_m}{\mathrm{d}t} v^1 \,\mathrm{d}x + a\left(\int_\Omega |u_m|^q \,\mathrm{d}x\right)\int_\Omega \nabla u_m \nabla v^1 \,\mathrm{d}x + b(l(u_m))\int_\Omega u_m v^1 \,\mathrm{d}x = \int_\Omega f(x, u_m) v^1 \,\mathrm{d}x,$$

a.e. $0 \leqslant t \leqslant T$, by (8) it follows that

$$\int_\Omega \frac{\mathrm{d}u_m}{\mathrm{d}t} v \,\mathrm{d}x = \int_\Omega \frac{\mathrm{d}u_m}{\mathrm{d}t} v^1 \,\mathrm{d}x =$$

$$\int_\Omega f(x, u_m) v^1 \,\mathrm{d}x - a\left(\int_\Omega |u_m|^q \,\mathrm{d}x\right)\int_\Omega \nabla u_m \nabla v^1 \,\mathrm{d}x - b(l(u_m))\int_\Omega u_m v^1 \,\mathrm{d}x.$$

Consequently

$$|\langle u'_m, v\rangle| \leqslant C_3\left(M\|u_m\|_{H_0^1(\Omega)} + \left(c|\Omega|^{\frac{1}{2}} + \beta\right)\|u_m\|_{L^2(\Omega)}\right) + c|\Omega|,$$

thus

$$\|u'_m\|_{H^{-1}} \leqslant C_3(M\|u_m\|_{H_0^1(\Omega)} + (c|\Omega|^{\frac{1}{2}} + \beta)\|u_m\|_{L^2(\Omega)}) + c|\Omega|, \tag{14}$$

since $\|v\|_{H_0^1} \leqslant 1$. Integrate (14) from 0 to T and (12) we obtain

$$\|u'_m\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C_4. \tag{15}$$

Next we pass to limits in (9) as $m \to \infty$. According to (13), (15), we see that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{u'_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. We deduce the existence a subsequence of (u_m) such that

$$u_m \rightharpoonup u \text{ in } L^2(0,T;H_0^1(\Omega)),\quad u'_m \rightharpoonup u' \text{ in } L^2(0,T;H^{-1}(\Omega)),$$
$$u_m \to u \text{ in } L^2(0,T;L^p(\Omega)),\quad 1 \leqslant p < 2N/(N-2),$$

passing to limits in (9) we get that

$$\int_\Omega u' w_k + a\left(\int_\Omega |u|^q dx\right)\int_\Omega \nabla u \nabla w_k dx + b(l(u))\int_\Omega u w_k dx = \int_\Omega f(x,u) w_k dx. \tag{16}$$

From the completeness of w_k, (16) holds with w_k replaced by any $v \in H_0^1(\Omega)$. Hence

$$\int_\Omega u' v dx + a\left(\int_\Omega |u|^q dx\right)\int_\Omega \nabla u \nabla v dx + b(l(u))\int_\Omega u v dx = \int_\Omega f(x,u) v dx.$$

By Lemma 1, it shows u is a positive solution.

[References]

[1] Corr a F J S A. On positive solutions of nonlocal and nonvariational elliptic problems[J]. Nonlinear Anal, 2004, 59(7): 1 147-1 155.

[2] Ma T F. Remarks on an elliptic equation of Kirchhoff type[J]. Nonlinear Anal, 2005, 63(5): e(1 967)-e(1 977).

[3] Corr a F J S A, Menez S D B, Ferreira J. On a class of problems involving a nonlocal operator[J]. Applied Math Com, 2004, 147(2): 475-489.

[4] Chipot M. Some remarks on nonlocal elliptic and parabolic problems[J]. Nonlinear Anal, 1997, 30(4): 4 619-4 627.

[5] Sta zy R. Nonlocal elliptic equations[J]. Nonlinear Anal, 2001, 47(5): 3 579-3 584.

[6] Corr a F J S A, Danielc de Morais Filho. On a class of nonlocal elliptic problems via Galerkin method[J]. J Math Anal Appl, 2005, 310(1): 177-187.

[7] Chang N H, Chipot M. On some model diffusion problems with a nonlocal lower order terms[J]. Chin Ann Math, 2003, 24B(2): 147-166.

[8] Corr a F J S A, Menez S D B. Existence of solution to nonlocal and singular elliptic problems via Galerkin method[J]. Electron J Differental Equation, 2004, 19: 1-10.

[9] Abderrahmane, Hachimi, Mouldy Rchid Sidi Ammi. Thermistor problem: A nonlocal parabolic problem[J]. Elec J Differental Equations Conference, 2004, 11: 117-128.

[10] Delmling K. Nonlinear Functional Analysis[M]. Berlin: Springer-Verlag, 1985.

[责任编辑: 丁　蓉]