Multigrid Methods for Mortar-Type Nonconforming Quadrilateral Element

Wang Feng[1], Xu Wei[2]

(1. School of Mathematics and Computer Science, Nanjing Normal University, Nanjing 210097, China)
(2. School of Sciences, PLA University of Science and Technology, Nanjing 211101, China)

Abstract: Multigrid algorithms for mortar-type nonconforming quadrilateral element were discussed. An intergrid transfer operator were proposed for the nonested mortar element spaces. It was proved that the W-cycle and variable V-cycle multigrid methods were both optimal. And the numerical experiments confirmed our results.

Key words: multigrid method, mortar element, nonconforming quadrilateral element
CLC number: O242.2 **Document code**: A **Article ID**: 1001-4616(2008)03-0016-08

Mortar型非协调四边形元多重网格方法

王 锋[1], 徐 为[2]

(1. 南京师范大学数学与计算机科学学院, 江苏 南京, 210097)
(2. 解放军理工大学理学院, 江苏 南京, 211101)

[摘要] 讨论了 Mortar型四边形元的多重网格方法. 针对非嵌套的 Mortar元空间, 提出了一种网格转移算子, 并证明了 W 循环和可变的 V循环多重网格方法是最优的. 数值实验验证了我们的理论结果.

[关键词] 多重网格方法, Mortar型有限元, 非协调四边形元

Mortar element method is a nonconforming domain decomposition method with nonoverlapping subdomains. The general concept for mortar techniques was originally introduced in [1] for coupling spectral element methods and finite element methods. The meshes on different subdomains need not align across subdomain interfaces, and the matching condition on the interfaces is only enforced weakly by integral conditions. However, most of the works use conforming finite element spaces in each subdomain. Marcinkowski first constructed and analyzed the mortar method for second order elliptic problems with locally nonconforming P_1 element[2]. Chen and Kim et al. considered mortar type quadrilateral elements, respectively[3,4].

Nonconforming quadrilateral element is a new element introduced by Douglas et al[5,6]. This element is a modification of the rotated bilinear elements[7], preserving a certain canceling property. Based on these elements, Cai et al[8] proposed stabilized Stokes elements. Such quadrilateral elements were used to define a nonconforming mixed element for full Maxwell equations[9], which is the first nonconforming element for Maxwell equations. These elements have also been used to planar linear elasticity problems[10].

In this paper, we study multigrid algorithms for the mortar-type nonconforming quadrilateral element for second order elliptic problems. An intergrid transfer operator is presented for the nonested mortar element spaces. With the framework developed in [11], we prove that the W-cycle method is optimal, i.e., the convergence rate

Received date: 2007-07-12.
Foundation item: Supported by the NSF of Jiangsu Province (BK2006215).
Corresponding author: Wang Feng, doctor, majored in finite element theory. E-mail: fengwang@live.cn

is independent of the level, and the variable V-cycle algorithm provides a preconditioner with a condition number which is bounded independently of the levels. Numerical experiments demonstrate that the optimal convergence property for the W-cycle algorithm holds with any number of smoothing steps.

For conveniences, we denote by C a universal constant which is independent of the mesh size and level but whose values can differ from place to place throughout this paper.

1 Mortar Nonconforming Quadrilateral Element

In this chapter, we deal with the following model problem:

$$-\triangle u=f, \quad \text{in } \Omega \tag{1}$$

with zero Dirichlet boundary condition, where $\Omega\in\mathbf{R}^2$ is a bounded rectangle or L-shaped domain, $f\in L^2(\Omega)$. The variational form of (1) is to find $u\in H_0^1(\Omega)$ such that

$$a(u, v)=(f, v), \quad \forall v\in H_0^1(\Omega), \tag{2}$$

where the bilinear form $a(u, v)=\int_\Omega \nabla u\cdot\nabla v\,dx$, $\forall u, v\in H^1(\Omega)$.

We assume that Ω is decomposed into nonoverlapping open rectangular subdomains Ω_k, $1\leqslant k\leqslant N$, $\Omega=\bigcup_{k=1}^N\Omega_k$ with $\Omega_i\cap\Omega_j=\emptyset$, $i\neq j$. Let T_1^k be the coarsest quasi-uniform triangulation of the domain Ω_k made of elements that are rectangles whose edges are parallel to x-axis or y-axis. The mesh parameter $h_{1,k}$ is the diameter of the largest element in T_1^k, and let $h_1=\max_{1\leqslant k\leqslant N}h_{1,k}$. Denote the global mesh $\bigcup_{k=1}^N T_1^k$ by T_1. We refine the triangulation T_1 to produce T_2 by splitting each rectangle of T_1 into four rectangles by joining the opposite midpoints of the edges of the rectangle. The triangulation T_2 is quasi-uniform of size $h_2=h_1/2$. Repeating this process, we get the l-time refined triangulation T_l with mesh size $h_l=h_1 2^{1-l}$, $l=1, \cdots, L$. Let ε_l^k be the set of the edges of T_l^k. Denote the global edges $\bigcup_{k=1}^N\varepsilon_l^k$ by ε_l. Moreover, M_e is the midpoint of $e\in\varepsilon_l$. For each triangulation $T_l^k(\Omega_k)$, we introduce the nonconforming finite element employed in [5]. Let $\hat{K}=[-1, 1]^2$ be the reference square in $\mathbf{R}^2$ and $\hat{Q}=\mathrm{Span}\{1, x, y, (x^2-\frac{5}{3}x^4)-(y^2-\frac{5}{3}y^4)\}$ be the reference finite element space on $\hat{K}$. Define a finite dimensional space by

$$V_l^k(\Omega_k)=\{v\in L^2(\Omega_k)\,|\,v|_K=\hat{v}\circ F_K^{-1},\ \hat{v}\in\hat{Q},\ \forall K\in T_l^k,\ v \text{ is continuous at } M_e,\ \forall e\in\varepsilon_l(\Omega_k),$$
$$\text{and } v|_K(M_e)=0,\ \forall e\in\varepsilon_l(\partial\Omega_k\cap\partial\Omega)\},$$

where $F_K: \hat{K}\to K$ is an affine mapping for all $K\in T_l^k$. Note that for $v\in V_l^k(\Omega_k)$, $\int_e v\,dx=|e|\cdot v(M_e)$, $\forall e\in\varepsilon_l(\Omega_k)$. Let $V_l=\prod_{k=1}^N V_l^k=\{v_l\,|\,v_l^k=v_l|_{\Omega_k}\in V_l^k(\Omega_k)\}$. Obviously, we have $V_1\not\subset V_2\not\subset\cdots\not\subset V_L$.

We restrict ourselves to the geometrically conforming situation where the intersection between the boundary of any two different subdomains $\Omega_i\cap\Omega_j$, $i\neq j$, is either empty set, a vertex, or a common edge. Let Γ_{ij} denote the open edge that is common to two subdomains Ω_i and Ω_j. Further Γ_{ij} will be also called interface and denoted by γ_m $(1\leqslant m\leqslant M)$. We also need to introduce a global interface Γ as the union of all interfaces between the subdomains, i.e., $\Gamma=\bigcup_{k=1}^N\overline{\partial\Omega_i\setminus\partial\Omega}=\bigcup_{ij}\Gamma_{ij}=\bigcup_{m=1}^M\gamma_m$. Note that each edge Γ_{ij} inherits two triangulations made of segments that are edges of elements of the triangulation of Ω_i and Ω_j, respectively. In this way, each Γ_{ij} is provided with two independent and different 1D meshes which are denoted by $T_l^i(\Gamma_{ij})$ and $T_l^j(\Gamma_{ij})$. Because our solution space is not contained in $H_0^1(\Omega)$, we have to introduce some matching conditions over all interfaces $\Gamma_{ij}\subset\Gamma$ which are sufficient to ensure the optimality of the global approximation. One of the sides of Γ_{ij} is defined as a mortar (master) one, denoted by $\gamma_{m(i)}$, and the other as a nonmortar (slave) one denoted by $\delta_{m(j)}$. Let the mortar side of Γ_{ij} be chosen by the condition: $h_{l,i}\leqslant h_{l,j}$. An auxiliary test space $M_l(\delta_{m(j)})$ is defined by

$$M_l(\delta_{m(j)})=\{v\in L^2(\delta_{m(j)})\,|\,v|_e=\text{constant},\ \forall e\in T_l(\delta_{m(j)}):=T_l^j(\delta_j)\}.$$

The dimension of $M_l(\delta_{m(j)})$ is equal to the number of elements on $\delta_{m(j)}$. For each nonmortar edge $\delta_{m(j)} = \Gamma_{ij} \subset \Gamma$, we introduce the L^2 - orthogonal projection operator $Q_{l\,\delta_{m(j)}}: L^2(\Gamma_{ij}) \to M_l(\delta_{m(j)})$ by $(Q_{l\,\delta_{m(j)}} v, w)_{L^2(\delta_{m(j)})} = (v, w)_{L^2(\delta_{m(j)})}$, for all $w \in M_l(\delta_{m(j)})$, where $(\cdot, \cdot)_{L^2(\delta_{m(j)})}$ denotes the L^2 inner product over $L^2(\delta_{m(j)})$.

We now define the following mortar finite element space for nonconforming quadrilateral elements:

$$V_l = \{v \in V_l \mid \forall\ \gamma_{m(i)} = \delta_{m(j)} \subset \Gamma,\ Q_{l\,\delta_{m(j)}}(v|_{\gamma_{m(i)}}) = Q_{l\,\delta_{m(j)}}(v|_{\delta_{m(j)}})\}.$$

The condition of the equality of the L^2 - orthogonal projection of the traces onto the test space for each interface is called the mortar condition and can be equivalently rewritten as

$$\int_{\delta_{m(j)}} (v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}) w \, ds = 0, \quad \forall w \in M_l(\delta_{m(j)}). \tag{3}$$

It is worth to note that $V_l \not\subset H_0^1(\Omega)$.

Since functions in our discrete space V_l are not continuous, we must use a modified variational form $a_l(\cdot, \cdot)$ in the discretized problem. Define

$$a_{l\,k}(u_l^k, v_l^k) = \sum_{E \in T_l^k} \int_E \nabla u_l^k \cdot \nabla v_l^k \, dx, \quad \forall u_l^k, v_l^k \in V_l^k,$$

$$a_l(u_l, v_l) = \sum_{k=1}^{N} a_{l,k}(u_l, v_l), \quad \forall u_l, v_l \in V_l. \tag{4}$$

Obviously, the form $a_l(\cdot, \cdot)$ is positive-definite on V_l. For any $v_l \in V_l$, we introduce a broken energy norm: $\| v_l \|_l = \left(\sum_{k=1}^{N} \| v_l \|_{l,k}^2 \right)^{1/2}$, where $\| v_l \|_{l\,k}^2 = a_{l\,k}(v_l, v_l)$.

The mortar-type nonconforming quadrilateral element approximation of (2) is to find $u_l \in V_l$ such that

$$a_l(u_l, v_l) = (f, v_l), \quad \forall\ v_l \in V_l. \tag{5}$$

The following error estimate can be found in [4].

Theorem 1 Let u, u_l be the solutions of (2), (5) respectively, then

$$\| u - u_l \|_l^2 \leqslant C \sum_{k=1}^{N} h_{l\,k}^2 \, | u |_{2,k}^2. \tag{6}$$

2 Multigrid Algorithm

We now apply the theory developed in [11] to construct our multigrid algorithm. Before giving the algorithm, we must define a suitable intergrid transfer operator for the nonnested spaces V_l. First we define an operator $J_l^k: V_{l-1}^k \to V_l^k$ as follows

$$(J_l^k v)(M_e) = \begin{cases} 0, & e \subset \partial\Omega_k \cap \Omega, \\ v(M_e), & e \subset \partial\Omega_k \backslash \partial\Omega, \text{ or } e \not\subset E,\ E \in T_{l-1}^k, \\ \dfrac{1}{2}(v|_{E_1}(M_e) + v|_{E_2}(M_e)), & e \subset \partial E_1 \cap \partial E_2,\ E_1, E_2 \in T_{l-1}^k, \end{cases} \tag{7}$$

where e is an edge of E and $E \in T_l^k$.

Before we start with the investigation of the operator J_l^k, it will be useful to collect some formulas. For $E \in T_l^k$, $v \in V_{l-1}^k$, define $b_E^i = v|_E(M_{e_i})$ (see Fig. 1 for the notation), and set

$$s_E = b_E^1 + b_E^2 + b_E^3 + b_E^4, \quad \triangle_E^1 = b_E^4 - b_E^2,$$

$$\theta_E^0 = b_E^2 + b_E^4 - b_E^1 - b_E^3, \quad \triangle_E^2 = b_E^3 - b_E^1.$$

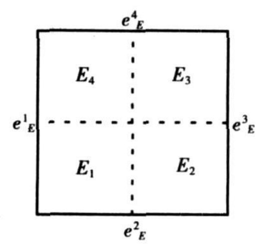

Fig.1 Edges and subsquares of E in T_{l-1}

Then, with the subscript E omitted, we have the next lemma.

Lemma 1 It holds that

$$\| v \|_{L^2(E)}^2 = h_{l-1,k}^2 \left(\frac{s^2}{16} + \frac{(\triangle^1)^2}{12} + \frac{(\triangle^2)^2}{12} + \frac{23}{2\,520}(\theta^0)^2 \right), \tag{8}$$

$$\|\nabla v\|_{L^2(E)}^2 = (\Delta^1)^2 + (\Delta^2)^2 + \frac{37}{14}(\theta^0)^2, \tag{9}$$

$$\|\nabla v\|_{L^2(E_i)}^2 \leqslant \frac{3}{4}(\Delta^1)^2 + \frac{3}{4}(\Delta^2)^2 + \frac{93}{56}(\theta^0)^2, \tag{10}$$

$$\frac{23h_{l-1,k}^2}{630}\sum_{i=1}^{4}(b^i)^2 \leqslant \|v\|_{L^2(E)}^2 \leqslant \frac{h_{l-1,k}^2}{4}\sum_{i=1}^{4}(b^i)^2. \tag{11}$$

Proof From the similar argument in [12], we can see it is sufficient to prove (8) − (11) for the master square $E = (-1, 1)^2$. A straightforward calculation gives

$$v = v(x, y) = \frac{1}{4}s + \frac{\Delta^2}{2}x + \frac{\Delta^1}{2}y + \frac{3}{8}\theta^0\left[\left(x^2 - \frac{5}{3}x^4\right) - \left(y^2 - \frac{5}{3}y^4\right)\right]. \tag{12}$$

Now direct integration yields the desired results in (8), (9). Also, (11) follows from (8) by computing the eigenvalues of the symmetric 4×4 matrix $\boldsymbol{T}^t\boldsymbol{D}\boldsymbol{T}$, where $D = \mathrm{diag}(1/16, 1/12, 1/12, 23/2\,520)$, $\boldsymbol{T}$ stands for the transformation matrix from the vector (b^1, b^2, b^3, b^4) to $(s, \Delta^1, \Delta^2, \theta^0)$, and $\boldsymbol{T}^t$ is the transpose of $\boldsymbol{T}$. These eigenvalues are $23/630$, $1/6$, $1/6$ and $1/4$, which implies (11).

(10) can be get by a straight forward calculation and triangle inequality.

Lemma 2 For any $v \in V_{l-1}^k$, it holds that

$$\|J_l^k v\|_{0,k} \leqslant C\|v\|_{0,k}, \tag{13}$$

$$\|J_l^k v - v\|_{0,k} \leqslant Ch_{l,k}\|v\|_{l-1,k}, \tag{14}$$

$$\|J_l^k v\|_{l,k} \leqslant C\|v\|_{l-1,k}, \tag{15}$$

$$\|J_l^k v - v\|_{0,\Gamma_m} \leqslant Ch_{l,k}^{1/2}\|v\|_{l-1,k}. \tag{16}$$

Proof From (11) in Lemma 1, we can get (13) the L^2 stability of J_l^k as

$$\|J_l^k v\|_{0,k}^2 \leqslant \sum_{E\in T_l^k}\frac{h_l^2}{4}\sum_{i=1}^{4}(b^i)^2 \leqslant \frac{h_l^2}{4}\sum_{E\in T_{l-1}^k}\frac{8\,275}{1\,024}\sum_{i=1}^{4}(b^i)^2 \leqslant 14\|v\|_{0,k}^2. \tag{17}$$

To obtain (14), we need to deal with the difference $J_l^k v - v$ which is an element of $V_{l-1}^k + V_l^k$. From the definition (7), we can get two cases of value on the middle point of edges for $J_l^k v - v$. If an edge $e \in \varepsilon_l^k$ belongs to the interior of some square in T_{l-1}^k or be contained in $\partial\Omega_k \backslash \partial\Omega$, then $J_l^k v - v$ vanishes on M_e. What remains are edges e that belong to either a (boundary or interior) edge of the partition T_{l-1}^k or $\partial\Omega$. We can get $(J_l^k v - v)(M_e)$ exactly. Taking $e = \overline{P_1M_1} \subset \partial E_1$ for example (see Fig. 2),

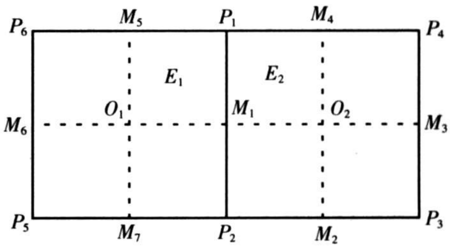

Fig.2 A illustration for Lemma 2

$$\begin{aligned}(J_l^k v - v)|_{E_1}(M_e) &= -\frac{7}{256}(v(M_3) - v(M_6)) - \frac{39}{256}(v(M_2) - v(M_7)) + \frac{25}{256}(v(M_4) - v(M_5)) = \\ &-\frac{7}{256}(v(M_3) - v(M_1)) - \frac{7}{256}(v(M_1) - v(M_6)) - \frac{39}{256}(v(M_2) - v(M_1)) \\ &-\frac{39}{256}(v(M_1) - v(M_7)) + \frac{25}{256}(v(M_4) - v(M_1)) + \frac{25}{256}(v(M_1) - v(M_5)).\end{aligned}$$

Note that for the edge $e \subset \partial\Omega_k \cap \partial\Omega$ of E_1, $(J_l^k v - v)|_{E_1}(M_e)$ is similar to express as above. Using the norm equivalence, we can obtain $[(J_l^k v - v)|_{E_1}(M_e)]^2 \leqslant C(|v|_{H^1(E_1)}^2 + |v|_{H^1(E_2)}^2)$, where $E_1, E_2 \in T_{l-1}^k$. So we have $\|J_l^k v - v\|_{0,k}^2 \leqslant Ch_{l,k}^2\sum_{E\in T_l^k}\sum_{e\subset\partial E}[(J_l^k v - v)(M_e)]^2 \leqslant C\,h_{l,k}^2\|v\|_{l-1,k}^2$, which gives (14).

(15) follows from the same argument as in the proof of (14). We refer to [13] for the proof of (16).

Based on the operator J_l^k, we define an intergrid transfer operator $J_l: V_{l-1} \to V_l$ as follows: for any $v = (v^1, v^2, \ldots, v^N) \in V_{l-1}$, $J_l v = (J_l^1 v^1, J_l^2 v^2, \ldots, J_l^N v^N) \in V_l$. Define the operator $\Xi_{l,\delta_{m(j)}}: V_l \to V_l$ by

$$(\Xi_{l,\delta_{m(j)}}(v))(M_e) = \begin{cases}(Q_{l,\delta_{m(j)}}(v|_{\Gamma_{m(i)}} - v|_{\delta_{m(j)}}))(M_e), & e \in T_l(\delta_{m(j)}),\\ 0, & \text{otherwise}\end{cases} \tag{18}$$

Then for any $v\in V_l$, let $v^* = v+\sum_{m=1}^{M}\Xi_{l\,\delta_{m(j)}}v$, and it is easy to see $v^*\in V_l$.

After the above preparation, we define an intergrid transfer operator $I_l: V_{l-1}\to V_l$:

$$I_l v = J_l v+\sum_{m=1}^{M}\Xi_{l\,\delta_{m(j)}}(J_l v)\in V_l,\quad v\in V_{l-1} \tag{19}$$

We now introduce some useful operators: A_l denotes the operator on V_l, which is defined by

$$(A_l u, v) = a_l(u, v),\quad \forall u, v\in V_l. \tag{20}$$

In terms of the operator A_l, the discrete problem (5) can be written as

$$A_l u_l = f_l, \tag{21}$$

where $f_l\in V_l$ satisfies $(f_l, v_l)=f(v_l)$, $\forall\ v_l\in V_l$.

Based on I_l, we define the projection operator $P_{l-1}: V_l\to V_{l-1}$ and $P^0_{l-1}: V_l\to V_{l-1}$ by

$$a_{l-1}(P_{l-1}u, v) = a_l(u, I_l v),\quad u\in V_l,\quad \forall v\in V_{l-1},\quad l=2, \cdots, L, \tag{22}$$

$$(P^0_{l-1}u, v) = (u, I_l v),\quad u\in V_l, \forall v\in V_{l-1},\quad l=2, \cdots, L \tag{23}$$

In order to define multigrid algorithm, we construct smoothing operators, including Gauss-Seidel and conjugate gradient iterations, which satisfy the following condition (see the proof of (A.4) in the chapter 5 of [14]):

(R) There exists a constant $C_R\geqslant 1$ independent of l such that

$$\frac{\|u\|_0^2}{\lambda_l}\leqslant C_R(R_l u, u),\quad \forall u\in V_l, \tag{24}$$

for both $R_l=(I-K^*_l K_l)A_l^{-1}$ and $R_l=(I-K_l K^*_l)A_l^{-1}$, where $K_l=I-R_l A_l$, $K^*_l=I-R^t_l A_l$, R^t_l be the adjoint of R_l with respect to $(\cdot, \cdot)$, and λ_l is the maximum eigenvalue of A_l.

A general multigrid operator $B_l: V_l\to V_l$ can be defined recursively[11].

According to [11], an key condition in the multigrid analysis is the regularity and approximation assumption:

(H1) Let λ_l be the maximum eigenvalue of A_l. There exists an $\alpha\in[0, 1]$ such that

$$|a_l(v-I_l P_{l-1}v, v)|\leqslant C\left(\frac{\|A_l v\|_0^2}{\lambda_l}\right)^{\alpha}a_l(v, v)^{1-\alpha},\quad \forall v\in V_l. \tag{25}$$

The convergence rate for the above multigrid algorithm is measured by a convergence factor δ_l satisfying

$$|a_l((I-B_l A_l)v, v)|\leqslant \delta_l\, a_l(v, v). \tag{26}$$

Following [11], we state two propositions.

Proposition 1 (W-cycle) Under the condition (H1), if $p=2$ and $m(l)=m$ is large enough, then the convergence factor in (26) is

$$\delta_l=\frac{C}{C+m^{\alpha}}. \tag{27}$$

Proposition 2 (variable V-cycle preconditioner) Assume that (H1) is valid and the number of smoothing $m(l)$ increases as l decreases in such a way that $\beta_0 m(l)\leqslant m(l-1)\leqslant\beta_1 m(l)$, holds with $1<\beta_0\leqslant\beta_1$. Then there exists an $M>0$ independent of l such that

$$C_0^{-1}a_l(v, v)\leqslant a_l(B_l A_l v, v)\leqslant C_0 a_l(v, v),\quad \forall v\in V_l, \tag{28}$$

with $C_0=\frac{M+m(l)^{\alpha}}{m(l)^{\alpha}}$.

3 Verification of Assumption (H1)

From section 2, we know that if we can prove that the assumption (H1) holds for the mortar element method for nonconforming quadrilateral element, then convergence results of our multigrid methods are obtained. To this end, in the following subsection, we first give some technical lemmas.

Lemma 3 For any $v\in V_{l-1}$, it holds that

(1) $\| I_l v \|_l \leqslant C \| v \|_{l-1}$, (2) $\| v - I_l v \|_0 \leqslant C h_l \| v \|_{l-1}$. (29)

Proof For the proof, we refer to Lemma 6 in [15].

Let $\Theta_l^k: H^2(\Omega_k) \to V_l^k(\Omega_k)$ be an interpolation operator defined as follows

$$(\Theta_l^k v)(M_e) = v(M_e), \quad \forall\ v \in H^2(\Omega_k),\ e \in \varepsilon_l^k. \tag{30}$$

Lemma 4 For the interpolation Θ_l^k and the transfer operator J_l^k, we have for any $v \in H^2(\Omega_k)$,

$$\| v - \Theta_l^k v \|_{0,k} + h_{l,k} \| v - \Theta_l^k v \|_{l,k} \leqslant C h_{l,k}^2 \| v \|_{2,k}, \tag{31}$$

$$\| v - J_l^k(\Theta_{l-1}^k v) \|_{0,k} + h_{l,k} \| v - J_l^k(\Theta_{l-1}^k v) \|_{l,k} \leqslant C h_{l,k}^2 \| v \|_{2,k}. \tag{32}$$

Proof The proof of (31) can be found in [5], here we only prove (32). Using (31) we get

$$\| v - J_l^k(\Theta_{l-1}^k v) \|_{0,k} \leqslant \| v - \Theta_l^k v \|_{0,k} + \| \Theta_l^k v - J_l^k(\Theta_{l-1}^k v) \|_{0,k} \leqslant$$
$$C h_{l,k}^2 \| v \|_{2,k} + \| \Theta_l^k v - J_l^k(\Theta_{l-1}^k v) \|_{0,k}. \tag{33}$$

Observing the proof of (15) in Lemma 2, using homogeneity argument in [16], we have

$$\| \Theta_l^k v - J_l^k(\Theta_{l-1}^k v) \|_{0,k}^2 = \sum_{E \in T_l^k} \| \Theta_l^k v - J_l^k(\Theta_{l-1}^k v) \|_{L^2(E)}^2 \leqslant$$
$$C h_{l,k}^2 \sum_{E \in T_l^k} \| \hat{\Theta}_l^k v - J_l^k(\hat{\Theta}_{l-1}^k v) \|_{L^2(\hat{E})}^2 \leqslant$$
$$C h_{l,k}^2 \sum_{E \in T_l^k} \sum_{i=1}^{4} [(\hat{\Theta}_l^k v - J_l^k(\hat{\Theta}_{l-1}^k v))(M_{\hat{e}_i})]^2 =$$
$$C h_{l,k}^2 \sum_{E \in T_l^k} \{ -\frac{7}{256}(\hat{v}(M_3) - \hat{v}(M_6)) - \frac{39}{256}(\hat{v}(M_2) - \hat{v}(M_7)) + \frac{25}{256}(\hat{v}(M_4) - \hat{v}(M_5)) \}^2 \leqslant$$
$$C h_{l,k}^2 \sum_{E \in T_l^k} (\sup_{\hat{E}} |\hat{v}|)^2 \leqslant C h_{l,k}^2 \sum_{E \in T_l^k} |\hat{v}|_{H^2(\hat{E})}^2 \leqslant C h_{l,k}^4 |v|_{2,k}^2. \tag{34}$$

Combining (33) with (34) yields the first part of (32). The second part of (32) follows from the observation

$$\| v - J_l^k(\Theta_{l-1}^k v) \|_{l,k}^2 \leqslant C \sum_{E \in T_l^k} \sum_{i=1}^{4} [(v - J_l^k(\Theta_{l-1}^k v))(M_{e_i})]^2 \leqslant$$
$$C h_{l,k}^{-2} \| v - J_l^k(\Theta_{l-1}^k v) \|_{0,k}^2 \leqslant C h_{l,k}^2 \| v \|_{2,k}^2,$$

the proof is completed.

Based on the operator Θ_l^k, we now define $\Theta_l: H^2(\Omega) \cap H_0^1(\Omega) \to V_l$ as follows. For any $v \in H^2(\Omega) \cap H_0^1(\Omega)$, $\Theta_l v = (\Theta_l^1 v^1, \ldots, \Theta_l^N v^N) \in V_l$, where $v^k = v|_{\Omega_k}$,

$$\Pi_l \zeta = \Theta_l \zeta + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(\Theta_l \zeta). \tag{35}$$

Lemma 5 For the operator Π_l, we have

$$\| \zeta - \Pi_l \zeta \|_0 + h_l \| \zeta - \Pi_l \zeta \|_l \leqslant C h_l^2 \| \zeta \|_2, \quad \forall\ \zeta \in H^2(\Omega) \cap H_0^1(\Omega). \tag{36}$$

Proof The proof is similar as the one of Lemma 9 in [13], we omit here.

For the operator Π_l and the intergrid transfer operator I_l, the following lemma is crucial.

Lemma 6 For any $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\| \zeta - I_l \Pi_{l-1} \zeta \|_l \leqslant C h_l \| \zeta \|_2. \tag{37}$$

Proof By the definition of Π_{l-1} and I_l, we obtain

$$\zeta - I_l \Pi_{l-1} \zeta = \zeta - I_l(\Theta_{l-1} \zeta) - \sum_{m=1}^{M} I_l(\zeta_{l-1,\delta_{m(j)}}(\Theta_{l-1} \zeta)) =$$
$$\zeta - J_l(\Theta_{l-1} \zeta) - \sum_{m=1}^{M} \zeta_{l,\delta_{m(j)}}(\Theta_{l-1} \zeta) - \sum_{m=1}^{M} I_l(\zeta_{l-1,\delta_{m(j)}}(\Theta_{l-1} \zeta)). \tag{38}$$

Using triangle inequality, we get

$$\| \zeta - I_l \Pi_{l-1} \zeta \|_l^2 \leqslant C(\| \zeta - J_l(\Theta_{l-1} \zeta) \|_l^2 + \sum_{m=1}^{M} \| \zeta_{l,\delta_{m(j)}}(\Theta_{l-1} \zeta) \|_l^2 +$$
$$\sum_{m=1}^{M} \| I_l(\zeta_{l-1,\delta_{m(j)}}(\Theta_{l-1} \zeta)) \|_l^2). \tag{39}$$

Using the similar arguments of the proof of Lemma 10 in [13] we complete the proof.

Lemma 7 For the operator P_{l-1}, we have

$$\| v - P_{l-1} v \|_0 \leqslant C h_l \| v \|_l, \quad \forall\ v \in V_l. \tag{40}$$

Proof The proof can be done by using dual argument, Lemma 5, 6, we refer to [15] for details.

Theorem 2 (H1) is valid with $\alpha = 1/2$, i.e.,

$$| a_l((I - I_l P_{l-1}) v, v) | \leqslant C \left(\frac{\| A_l v \|_0^2}{\lambda} \right)^{1/2} a_l(v, v)^{1/2}, \quad \forall v \in V_l. \tag{41}$$

Proof Using Lemma 3 and 7, arguing as Theorem 4.1 in [15], we can prove (41) easily.

4 Numerical Experiments

In this section, we present some numerical results to illustrate the theory developed in the earlier sections. These examples deal with the Poisson equation on the unit square with zero dirichlete boundary condition.

For simplicity, we decompose Ω into two subdomains: $\Omega_1 = (0, 1) \times (0, \frac{1}{2})$ as mortar domain, and $\Omega_2 = (0, 1) \times (\frac{1}{2}, 1)$ as nonmortar domain. The sizes of the coarsest grid are denoted by $h_{1,1}$ and $h_{1,2}$ respectively. Here we use Gauss-Seidel smoothing iteration. In our first experiments, we consider the W-cycle Multigrid Algorithms. Table 1 shows the number of iterations required to achieve the error reduction 10^{-3}, where the initial value is zero. l denotes the level number, d.o.f denotes the degrees of the freedom on each level. And $m(l)$ is the pre-smoothing steps and the post-smoothing steps. The iteration numbers are denoted by $\text{iter}_{(m(l), m(l))}$. The second test concerns the variable V-cycle Multigrid Algorithm. The results are shown in Table 2. We use the preconditioned conjugate gradient method for the discretized systems $A_l u_l = f_l$, $l = 2, 3, \cdots, L$, with the variable V-cycle preconditioners B_l. We denote $\text{iter}_{h\frac{1}{2}}$, $\text{iter}_{h\frac{2}{3}}$ the iterative numbers of PCG method, where the relative error of residue is less than 10^{-3} with $h_{l,1}/h_{l,2} = 1/2$ and $h_{l,1}/h_{l,2} = 2/3$ respectively. We also choose zero as initial value.

Table 1 Iterative numbers for the W-cycle with $h_{1,1} = \frac{1}{4}$, $h_{1,2} = \frac{1}{2}$ (left) and $h_{1,1} = \frac{1}{6}$, $h_{1,2} = \frac{1}{4}$ (right)

	$h_m : h_s = 2:1$					$h_m : h_s = 2:1$				
l	2	3	4	5	6	2	3	4	5	6
d.o.f	94	348	1 336	5 232	20 704	230	876	3 416	13 488	53 600
$\text{iter}_{(1,1)}$	7	8	8	8	8	8	10	10	10	10
$\text{iter}_{(2,2)}$	6	6	7	7	7	7	8	8	8	8
$\text{iter}_{(3,3)}$	5	5	5	5	5	6	6	6	6	6

Table 2 Iterative numbers for the PCG-method with $m(l) = 2^{L-l}$

l	2	3	4	5	6
$\text{iter}_{h\frac{1}{2}}$	2	2	3	3	3
$\text{iter}_{h\frac{2}{3}}$	2	2	2	2	3

From Table 1, we can see that the iteration numbers for the W-cycle multigrid algorithm remain constant with any number of smoothing steps when the mesh size decreases, which is better than Proposition 1. And experimental results form Table 2 show that the iterative numbers are independent of the meshes, so we state that the condition number of $B_l A_l$ is bounded uniformly.

[References]

[1] Bernardi C, Maday Y, Patera A T. Domain decomposition by the mortar element method[M] // Kaper H G, Garbey M, Pieper G W. Asymptotic and Numerical Methods for Partial Differential Equations with Critical Parameters. Dordrecht: Kluwer Academic Publishers, 1993: 269-286.

[2] Marcinkowski L. The mortar element method with locally nonconforming elements[J]. BIT, 1999, 39(4): 716-739.

[3] Chen J R, Xu X J. The mortar element method for rotated Q_1 element[J]. J Comp Math, 2000, 20(3): 313-324.

[4] Kim K, Yi D, Lee S. Mortar method for nonconforming finite elements [J]. App Math Comp, 2005, 167(1): 650-669.

[5] Douglas Jr J, Santos J E, Sheen D, et al. Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems [J]. Math Mode and Numer Anul, 1999, 33(4): 747-770.

[6] Cai Z, Douglas Jr J, Santos J E, et al. Nonconforming quadrilateral finite elements: a correction [J]. Calcolo, 2000, 37(4): 253-254.

[7] Rannacher R, Turek S. Simple nonconforming quadrilateral Stokes element [J]. Numer Meth for PDEs, 1992, 8(2): 97-111.

[8] Cai Z, Douglas Jr J, Ye X. A stable nonconforming quadrilateral finite element method for the stationary Stokes and Navier-Stokes equations [J]. Calcolo, 1999, 36(4): 215-232.

[9] Douglas Jr J, Santos J E, Sheen D. A nonconforming mixed finite element method for maxwells' equations [J]. Math Models Meth App Sci, 2000, 10(4): 593-613.

[10] Lee C, Lee J, Sheen D. A locking-free nonconforming finite element method for planar linear elasticity [J]. Adv in Comp Math, 2003, 19(1/3): 277-291.

[11] Bramble J H, Pasciak J E, Xu J. The analysis of multigrid algorithms with nonested spaces and noninherited quadratic forms [J]. Math Comp, 1991, 56(193): 1-34.

[12] Chen Z X, Oswald P. Multigrid and multilevel methods for nonconforming Q_1 elements [J]. Math Comput, 1998, 67(222): 667-693.

[13] Huang P Q, Chen J R. Multigrid methods for mortr-type rotated Q_1 elements for second order elliptic problems [J]. Math Numer Sinica, 2008, 30(1): 1-16.

[14] Bramble J H. Multigrid Methods [M]. Boston: Pitman, 1993.

[15] Xu X J, Chen J R. Multigrid for the mortar element method for P_1 nonconforming element [J]. Numer Math, 2001, 88(2): 381-398.

[16] Brenner S C, Scott L R. The Mathematical Theory of Finite Element Methods [M]. New York: Springer-Verlag, 1994.

[责任编辑: 丁　蓉]