Balanced Judicious Partitions of $(k, k-1)$-Biregular Graphs

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Abstract Bollobás and Scott conjectured that every graph with m edges and minimum degree at least 2 has a balanced bipartition with at most $m/3$ edges in each vertex class. They proved that most regular graphs have a balanced partitions with less than $m/4$ edges in each vertex class. In this paper, balanced bipartitions of $(k, k-1)$-biregular graphs were considered, and it was proved that every $(k, k-1)$-biregular graph admits a balanced bipartition with about $m/4$ edges in each vertex class.

Key words judicious partition, balanced bipartition, $(k, k-1)$-biregular graph
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$(k, k-1)$-双正则图的平衡 Judicious Partitions

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[摘要] Bollobás和 Scott提出猜想: 任意一个边数为 m 且最小度大于 1的图存在顶点集的平衡二部划分使得每一部分点集的导出子图包含的边数不超过 $m/3$. Bollobás和 Scott证明了绝大部分正则图存在顶点集的平衡二部划分使得每一部分点集的导出子图包含的边数比 $m/4$小. 这里讨论 $(k, k-1)$-双正则图的平衡二部划分, 证明了每一个 $(k, k-1)$-双正则图存在平衡二部划分使得每一部分点集的导出子图包含的边数是 $m/4$左右.

[关键词] judicious partition, 平衡二部划分, $(k, k-1)$-双正则图

Let G be a simple graph, and $V(G)$ is the vertex set of G. Denote the number of edges in G by $e(G)$. For $X \subset V(G)$, we define $G[X]$ to be the subgraph of G induced by X, and set $e(X) = e(G[X])$. For disjoint subsets $X, Y \subset V(G)$, we write

$$e(X, Y) = |\{xy \in E(G) | x \in X, y \in Y\}|$$

for the number of edges between X and Y. A cut (V_1, V_2) of G is a bipartition $V(G) = V_1 \cup V_2$. The size of the cut (V_1, V_2) is $e(V_1, V_2)$. The Max Cut problem asks for the maximum size of a cut in a graph G or, equivalently, the minimum of $e(V_1) + e(V_2)$ over partitions $V(G) = V_1 \cup V_2$. Problems like this involve maximizing or minimizing a single quantity. The partitioning problem we shall discuss in this paper refers to maximizing or minimizing several quantities simultaneously. Such problems are known as judicious partitioning problems, introduced by Bollobás and Scott[1]. For example, given a graph G, ask for the minimum of $\max\{e(V_1), \cdots, e(V_k)\}$ over partitions $V(G) = \bigcup_{i=1}^{k} V_i$. This problem has been investigated in many papers, see [2] and [3], for instance. We say that a partition $V(G) = \bigcup_{i=1}^{k} V_i$ is balanced if $|V_1| \leqslant \cdots \leqslant |V_k| \leqslant |V_1| + 1$. Similar to judicious parti-

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tions, we can ask for the minimum of $\max\{e(V_1), \cdots, e(V_k)\}$ over balanced partitions $V(G)=\bigcup_{i=1}^{k} V_i$. In the following, we always assume $e(G)=m$. From $K_{1,n}$ we can see that we cannot do better than $m/k+O(1)$. But increasing the minimum degree should improve the constant. The following conjecture is proposed by Bollobás and Scott in [4].

Conjecture 1[4] Every graph with m edges and minimum degree at least 2 has a balanced bipartition with at most $m/3$ edges in each vertex class.

In [3], Bollobás and Scott discussed balanced bipartitions of regular graphs, and proved the following two theorems.

Theorem 1[3] Let $k\geqslant 3$ be an odd integer. Then every k-regular graph G has a balanced bipartition $V(G)=V_1\cup V_2$ such that

$$\max\{e(V_1), e(V_2)\}\leqslant\frac{k-1}{4k}m.$$

The extremal graphs are sK_{k+1}, for $s\geqslant 1$.

Theorem 2[3] Let $k\geqslant 2$ be an even integer, and G be a k-regular graph. Then G has a balanced bipartition $V(G)=V_1\cup V_2$ such that

(a) $\max\{e(V_1), e(V_2)\}\leqslant\frac{1}{4}\cdot\frac{k}{k+1}m$ when $|V(G)|$ is even, where the extremal graphs are of the form $2tK_{k+1}$, $t\geqslant 1$.

(b) $\max\{e(V_1), e(V_2)\}\leqslant\frac{1}{4}\cdot\frac{k}{k+1}m+\frac{k}{4}$ when $|V(G)|$ is odd, where the extremal graphs are of the form $(2t+1)K_{k+1}$, $t\geqslant 0$.

Denote by $d(v)$, the degree of v in G. A (k, d)-biregular graph G is one that for every $v\in V(G)$, $d(v)=k$ or d.

In this paper, we consider balanced judicious bipartitions of $(k, k-1)$-biregular graphs.

1 Main Results

In this section, we consider balanced bipartitions of $(k, k-1)$-biregular graphs. The following two theorems are our main results which show that every $(k, k-1)$-biregular graph admits a balanced bipartition such that the number of edges in each vertex class is about $\frac{m}{4}$.

Theorem 3 Let $k\geqslant 3$ be an odd integer, and G be a $(k, k-1)$-biregular graph. Assume G has n_1 vertices with degree k. Then, $V(G)$ has a balanced bipartition into V_1, V_2 such that

(a) $\max\{e(V_1), e(V_2)\}\leqslant\frac{m}{4}-\frac{n_1}{8}$ when $|V(G)|$ is even;

(b) $\max\{e(V_1), e(V_2)\}\leqslant\frac{m}{4}-\frac{n_1}{8}+\frac{k-1}{8}$ when $|V(G)|$ is odd.

Proof Let $|V(G)|=n$. Suppose $V(G)=V_1\cup V_2$ is a balanced bipartition with $e(V_1, V_2)$ maximum among such partitions. Assume, without loss of generality, that $e(V_1)\geqslant e(V_2)$. Let $n_{i,1}$ and $n_{i,2}$ be the number of vertices in V_i with degree k and $k-1$, respectively. Then by Handshaking Lemma,

$$2m=k(n_{1,1}+n_{2,1})+(k-1)(n_{1,2}+n_{2,2})=(k-1)\cdot n+n_1.$$

So

$$m=(k-1)\cdot\frac{n}{2}+\frac{n_1}{2}. \qquad (1)$$

First suppose that n is even, then $|V_1|=|V_2|=\frac{n}{2}$. If for all $v\in V_1$,

$$|N(v)\cap V_1|\leqslant|N(v)\cap V_2|,$$

noting that k is odd, $k-1$ is even, we have

$$2e(V_1)\leqslant\frac{k-1}{2}\cdot n_{1,1}+\frac{k-1}{2}\cdot n_{1,2}=\frac{k-1}{2}\cdot\frac{n}{2}.$$

Thus by (1)

$$e(V_1)\leqslant\frac{k-1}{4}\cdot\frac{n}{2}=\frac{m}{4}-\frac{n_1}{8}. \tag{2}$$

So we may assume that there is a vertex, say v_1, in V_1 such that $|N(v_1)\cap V_1|>|N(v_1)\cap V_2|$. Then we claim that for all $w\in V_2$,

$$|N(w)\cap V_1|>|N(w)\cap V_2|. \tag{3}$$

Otherwise if there is $v_2\in V_2$ such that $|N(v_2)\cap V_1|\leqslant|N(v_2)\cap V_2|$, then let $V_1'=(V_1\setminus\{v_1\})\cup\{v_2\}$, $V_2'=(V_2\setminus\{v_2\})\cup\{v_1\}$. We get a balanced bipartition $V_1'\cup V_2'$, and $e(V_1',V_2')\geqslant e(V_1,V_2)+(|N(v_1)\cap V_1|-|N(v_1)\cap V_2|)+(|N(v_2)\cap V_2|-|N(v_2)\cap V_1|)\geqslant e(V_1,V_2)+1$. (Note that v_1 and v_2 may be adjacent). This is a contradiction to the maximality of $e(V_1,V_2)$. By (3),

$$2e(V_2)\leqslant\frac{k-1}{2}\cdot n_{2,1}+\frac{k-1-2}{2}\cdot n_{2,2}=\frac{k-1}{2}\cdot\frac{n}{2}-n_{2,2}.$$

It follows that

$$e(V_2)\leqslant\frac{k-1}{4}\cdot\frac{n}{2}-\frac{n_{2,2}}{2}. \tag{4}$$

Since

$$2e(V_1)+e(V_1,V_2)=k\cdot n_{1,1}+(k-1)\cdot n_{1,2}=k\cdot\frac{n}{2}-n_{1,2}$$

and

$$2e(V_2)+e(V_1,V_2)=k\cdot n_{2,1}+(k-1)\cdot n_{2,2}=k\cdot\frac{n}{2}-n_{2,2},$$

$$e(V_1)-e(V_2)=\frac{n_{2,2}-n_{1,2}}{2}. \tag{5}$$

By (1), (4) and (5),

$$e(V_1)=e(V_2)+\frac{n_{2,2}-n_{1,2}}{2}\leqslant\frac{k-1}{4}\cdot\frac{n}{2}-\frac{n_{2,2}}{2}+\frac{n_{2,2}-n_{1,2}}{2}=\frac{k-1}{4}\cdot\frac{n}{2}-\frac{n_{1,2}}{2}\leqslant\frac{k-1}{4}\cdot\frac{n}{2}=\frac{m}{4}-\frac{n_1}{8}. \tag{6}$$

Therefore $e(V_1)\leqslant\frac{m}{4}-\frac{n_1}{8}$ in any case. And by our assumption, $e(V_1)\geqslant e(V_2)$. It follows that $\max\{e(V_1),e(V_2)\}\leqslant\frac{m}{4}-\frac{n_1}{8}$.

Now suppose that n is odd. If $|V_1|=\frac{n+1}{2}$ and so $|V_2|=\frac{n-1}{2}$, then for all $v\in V_1$, $|N(v)\cap V_1|\leqslant|N(v)\cap V_2|$. Or else, if there is a vertex, say v_1', in V_1 such that $|N(v_1')\cap V_1|>|N(v_1')\cap V_2|$, then we could increase the size of the cut by moving v_1' from V_1 to V_2. Similar to (2),

$$e(V_1)\leqslant\frac{k-1}{4}\cdot\frac{n+1}{2}=\frac{m}{4}-\frac{n_1}{8}+\frac{k-1}{8}. \tag{7}$$

If $|V_1|=\frac{n-1}{2}$ and so $|V_2|=\frac{n+1}{2}$, then $e(V_1)-e(V_2)=\frac{n_{2,2}-n_{1,2}}{2}-\frac{k}{2}$. We can assume that for all $w\in V_2$, $|N(w)\cap V_1|>|N(w)\cap V_2|$. Because if there is a vertex, say v_2', in V_2 such that $|N(v_2')\cap V_1|<|N(v_2')\cap V_2|$, then we could increase the size of the cut by moving v_2' from V_2 to V_1. And if there is a vertex, say v_2'', in V_2 such that $|N(v_2'')\cap V_1|=|N(v_2'')\cap V_2|$, then let $V_1''=V_1\cup\{v_2''\}$, $V_2''=V_2\setminus\{v_2''\}$, and we get a balanced bipartition in above case (Namely, $|V_1''|=\frac{n+1}{2}$, $|V_2''|=\frac{n-1}{2}$, $e(V_1'',V_2'')$ is maximum and still $e(V_1)\geqslant e(V_2)$.) Therefore similar to (6), we have

$$e(V_1)=e(V_2)+\frac{n_{2,2}-n_{1,2}}{2}-\frac{k}{2}\leqslant\frac{k-1}{4}\cdot\frac{n+1}{2}-\frac{n_{2,2}}{2}+\frac{n_{2,2}-n_{1,2}}{2}-\frac{k}{2}=$$
$$\frac{k-1}{4}\cdot\frac{n+1}{2}-\frac{n_{1,2}}{2}-\frac{k}{2}<\frac{k-1}{4}\cdot\frac{n+1}{2}=\frac{m}{4}-\frac{n_1}{8}+\frac{k-1}{8}. \tag{8}$$

So we end our proof

From Theorem 3(a), one can see that if $k\geqslant 3$ is odd, then every k-regular graph with n vertices admits a balanced bipartition into V_1, V_2 such that $\max\{e(V_1), e(V_2)\}\leqslant\frac{m}{4}-\frac{n}{8}=\frac{1}{4}\cdot\frac{k-1}{k}m$, since $2m=kn$. This is a result of Theorem 1. Now we consider when k is even.

Theorem 4 Let $k\geqslant 2$ be an even integer, and G be a $(k, k-1)$-biregular graph. Assume G has n_2 vertices with degree $k-1$. Then, $V(G)$ has a balanced bipartition into V_1, V_2 such that

(a) $\max\{e(V_1), e(V_2)\}\leqslant\frac{m}{4}+\frac{n_2}{8}$ when $|V(G)|$ is even;

(b) $\max\{e(V_1), e(V_2)\}\leqslant\frac{m}{4}+\frac{n_2}{8}+\frac{k}{8}$ when $|V(G)|$ is odd

Proof Let $|V(G)|=n$. Suppose $V(G)=V_1\cup V_2$ is a balanced bipartition with $e(V_1, V_2)$ maximum among such partitions. Assume, without loss of generality, that $e(V_1)\geqslant e(V_2)$. Let $n_{i,1}$ and $n_{i,2}$ be the number of vertices in V_i with degree k and $k-1$, respectively. Then by Handshaking Lemma,

$$2m=k(n_{1,1}+n_{2,1})+(k-1)(n_{1,2}+n_{2,2}).$$

So

$$m=k\cdot\frac{n}{2}-\frac{n_2}{2}=(k-2)\cdot\frac{n}{2}+n-\frac{n_2}{2}. \tag{9}$$

First suppose that n is even, then $|V_1|=|V_2|=\frac{n}{2}$. If for all $v\in V_1$,

$$|N(v)\cap V_1|<|N(v)\cap V_2|,$$

noting that k is even, $k-1$ is odd, we have

$$2e(V_1)\leqslant\frac{k-2}{2}\cdot n_{1,1}+\frac{k-2}{2}\cdot n_{1,2}=\frac{k-2}{2}\cdot\frac{n}{2}.$$

Thus by (9),

$$e(V_1)\leqslant\frac{k-2}{4}\cdot\frac{n}{2}=\frac{m}{4}-\frac{n}{4}+\frac{n_2}{8}<\frac{m}{4}+\frac{n_2}{8}. \tag{10}$$

So we may assume that there is a vertex, say v_1, in V_1 such that $|N(v)\cap V_1|\geqslant|N(v)\cap V_2|$. Then we claim that for all $w\in V_2$,

$$|N(w)\cap V_1|\geqslant|N(w)\cap V_2|. \tag{11}$$

Otherwise if there is $v_2\in V_2$ such that $|N(v_2)\cap V_1|<|N(v_2)\cap V_2|$, then let $V'_1=(V_1\setminus\{v_1\})\cup\{v_2\}$, $V'_2=(V_2\setminus\{v_2\})\cup\{v_1\}$. We get a balanced bipartition $V'_1\cup V'_2$, and $e(V'_1, V'_2)\geqslant e(V_1, V_2)+(|N(v_1)\cap V_1|-|N(v_1)\cap V_2|)+(|N(v_2)\cap V_2|-|N(v_2)\cap V_1|)\geqslant e(V_1, V_2)+1$. (Note that v_1 and v_2 may be adjacent) This is a contradiction to the maximality of $e(V_1, V_2)$. By (11),

$$2e(V_2)\leqslant\frac{k}{2}\cdot n_{2,1}+\frac{k-2}{2}\cdot n_{2,2}=\frac{k}{2}\cdot\frac{n}{2}-n_{2,2}.$$

It follows that

$$e(V_2)\leqslant\frac{k}{4}\cdot\frac{n}{2}-\frac{n_{2,2}}{2}. \tag{12}$$

Since

$$2e(V_1)+e(V_1, V_2)=k\cdot n_{1,1}+(k-1)\cdot n_{1,2}$$

and

$$2e(V_2)+e(V_1,V_2)=k\cdot n_{2,1}+(k-1)\cdot n_{2,2},$$

$$e(V_1)-e(V_2)=\frac{n_{2,2}-n_{1,2}}{2}. \tag{13}$$

By (9), (12) and (13),

$$e(V_1)=e(V_2)+\frac{n_{2,2}-n_{1,2}}{2}\leqslant\frac{k}{4}\cdot\frac{n}{2}-\frac{n_{2,2}}{2}+\frac{n_{2,2}-n_{1,2}}{2}=\frac{k}{4}\cdot\frac{n}{2}-\frac{n_{1,2}}{2}\leqslant\frac{k}{4}\cdot\frac{n}{2}=\frac{m}{4}+\frac{n_2}{8}. \tag{14}$$

Therefore $e(V_1)\leqslant\frac{m}{4}+\frac{n_2}{8}$ in any case. By our assumption, $e(V_1)\geqslant e(V_2)$. It follows that $\max\{e(V_1),e(V_2)\}\leqslant\frac{m}{4}+\frac{n_2}{8}$.

Now suppose that n is odd. If $|V_1|=\frac{n-1}{2}$ and so $|V_2|=\frac{n+1}{2}$, then $e(V_1)-e(V_2)=\frac{n_{2,2}-n_{1,2}}{2}-\frac{k}{2}$, and for all $w\in V_2$, $|N(w)\cap V_1|\geqslant|N(w)\cap V_2|$. Otherwise, if there is a vertex, say v'_2, in V_2 such that $|N(v'_2)\cap V_1|<|N(v'_2)\cap V_2|$, then we could increase the size of the cut by moving v'_2 from V_2 to V_1. Therefore similar to (12), we have

$$e(V_2)\leqslant\frac{k}{4}\cdot\frac{n+1}{2}-\frac{n_{2,2}}{2}.$$

So

$$e(V_1)=e(V_2)+\frac{n_{2,2}-n_{1,2}}{2}-\frac{k}{2}=\frac{k}{4}\cdot\frac{n+1}{2}-\frac{n_{1,2}}{2}-\frac{k}{2}\leqslant\frac{k}{4}\cdot\frac{n}{2}+\frac{k}{8}-\frac{4k}{8}=\frac{m}{4}+\frac{n_2}{8}-\frac{3k}{8}. \tag{15}$$

Similarly, if $|V_1|=\frac{n+1}{2}$ and so $|V_2|=\frac{n-1}{2}$, then for all $v\in V_1$, $|N(v)\cap V_1|\leqslant|N(v)\cap V_2|$. Thus

$$2e(V_1)\leqslant\frac{k}{2}\cdot n_{1,1}+\frac{k-2}{2}\cdot n_{1,2}=\frac{k}{2}\cdot\frac{n+1}{2}-n_{1,2}\leqslant\frac{k}{2}\cdot\frac{n}{2}+\frac{k}{4}.$$

It follows that

$$e(V_1)\leqslant\frac{k}{4}\cdot\frac{n}{2}+\frac{k}{8}=\frac{m}{4}+\frac{n_2}{8}+\frac{k}{8}, \tag{16}$$

which ends our proof.

By Theorem 4(b), when k is even, K_{k+1} admits a balanced bipartition with at most $\frac{k^2+2k}{8}$ edges in each vertex class. And Theorem 2(b) gives the same result, where K_{k+1} is an extremal graph.

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