Multiple Solutions for Elliptic Equation With Critical Sobolev-Hardy Exponent and Inhomogeneous Term

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Abstract Let $2^* = \frac{2(N+\alpha)}{(N-2+\beta)}$, $N\geqslant 3$ be the limiting Sobolev exponent and $\Omega\subset \mathbf{R}^N$ open bounded set. It is showed that for $f(x)\in H_\beta^{-1}$ satisfying a suitable condition and $f(x)\neq 0$, the weighed elliptic problem:

$$\begin{cases} -\operatorname{div}(|x|^\beta \nabla u) = |x|^\alpha u^{p^*-1} + \varepsilon f(x), & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

admits its two solutions $\underline{u}$ and u in $H_{0\,\beta}^{1,p}(\Omega)$. Also $\underline{u}\geqslant 0$ and $u\geqslant 0$ for $f(x)\geqslant 0$. Notice that, in general, this is not the case if $f(x)=0$.

Key words p-Laplace equation, critical exponent, best constant, Sobolev-Hardy inequality
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带非齐次项和 Sobolev-Hardy临界指数的奇异椭圆方程的多解

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[摘要]　设 $2^* = \frac{2(N+\alpha)}{N-2+\beta}$, $N\geqslant 3$, 是极限 Sobolev指数, $\Omega\subset\mathbf{R}^N$ 是 $\mathbf{R}^N$ 中的开子集. 在 $f(x)\in H_\beta^{-1}$满足合适的条件且 $f(x)\neq 0$下, 讨论了一个带非齐次项和 Sobolev-Hardy临界指数的含权的椭圆型问题:

$$\begin{cases} -\operatorname{div}(|x|^\beta \nabla u) = |x|^\alpha u^{p^*-1} + \varepsilon f(x), & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

存在两个解 $\underline{u}$和 u在 $H_{0\,\beta}^{1,p}(\Omega)$中，且有 $\underline{u}\geqslant 0$, $u\geqslant 0$对所有的 $f(x)\geqslant 0$. 值得注意的是，当 $f(x)=0$时一般不成立.

[关键词]　p-阶拉普拉斯方程, 临界指数, 最佳常数, Sobolev-Hardy不等式

Brezis and Nirenberg in [1] proved that the semilinear elliptic problem:

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}} + \lambda u = 0, & x\in\Omega,\\ u>0, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

has no nontrivial solution when $\lambda=0$ and Ω is a starshaped domain, and has a nontrivial solution when $\lambda\in(0,$

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λ_1), where λ_1 is the first eigenvalue of the positive operator $-\Delta$. Cao and Zhou in [2] proved that the following elliptic problem:

$$\begin{cases} -\Delta u = c_1 u^{\frac{N+2}{N-2}} + f(x, u) + h, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

has a nontrivial solution if $f(x, t) \in (\mathrm{E} \times [0, +\infty))$ satisfies

$$0 \leqslant f(x, t) \leqslant c_2 t^{(N+2)/(N-2)} + \lambda t, \ \forall x \in \Omega, t \geqslant 0,$$

where $\lambda \in [0, \lambda_1)$, and $c_2 > 0$ is some constant.

In this paper, we consider the solutions for the following weighed elliptic problem:

$$\begin{cases} -\mathrm{div}(|x|^{\beta} \nabla u) = |x|^{\alpha} u^{p^*-1} + \varepsilon f(x), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\varepsilon > 0$, Ω is a bounded domain with smooth boundary in $\mathbf{R}^N$ containing the origin and the parameters β, σ satisfy $p^* \geqslant 2$, $N + \alpha > 0$, $\frac{N+\alpha}{p^*} + 1 = \frac{N+\beta}{2}$ and $\frac{\beta}{2} \geqslant \frac{\alpha}{p^*}$. $p^* = \frac{2(N+\alpha)}{N-2+\beta}$ is the critical exponent of the embeddings from $H_{0,\beta}^{1,p}(\Omega)$ to $L^p(\Omega)$ and $H_{0,\beta}^{1,p}(\Omega)$ is a standard Sobolev space, $f(x) \in H_\beta^{-1}(\Omega)$ is some given function satisfying a suitable condition and $H_\beta^{-1}(\Omega)$ denotes the dual space of $H_{0,\beta}^{1,p}(\Omega)$.

Let $p^* > 2$, $f(x)$ satisfying $|x|^{-\sigma/q} f(x) \in L^{q/(q-1)}(\Omega)$, $f(x) > 0$, exists $q \in [2, p^*)$. σ satisfies

$$N + \sigma > 0, \frac{N+\sigma}{q} + 1 = \frac{N+\beta}{2}, \frac{\beta}{2} \geqslant \frac{\sigma}{q},$$

where

$$\| |x|^{-\sigma/q} f(x) |_{\frac{q}{q-1}} = \left\{ \int_\Omega (|x|^{-\sigma/q} f(x))^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}.$$

Refer to [3] we have

$$\left(\int_{\mathbf{R}^N} |x|^{\alpha} |u|^{p^*} \right)^{1/p^*} \leqslant S^{\frac{-p^*}{2}} \left(\int_{\mathbf{R}^N} |x|^{\beta} |\nabla u|^2 \right)^{1/2}, \ \forall u \in H_{0,\beta}^{1,p}(\mathbf{R}^N),$$

then

$$S = \inf_{u \in H_{0,\beta}^{1,p}(\mathbf{R}^N), u \neq 0} \frac{\int_{\mathbf{R}^N} |x|^{\beta} |\nabla u|^2}{\left(\int_{\mathbf{R}^N} |x|^{\alpha} |u|^{p^*} \right)^{2/p^*}},$$

where S is the best Sobolev constant.

Let

$$f(x) \in H_\beta^{-1}(\Omega), \quad f(x) \geqslant 0, \quad f(x) \neq 0 \text{ in } \Omega,$$

$$\varepsilon_0 = \frac{1}{2}\left[1 - \left(\frac{1}{2} \right)^{p^*-2} \right] \left[\frac{p^*}{2} S^{\frac{p^*}{2}} \right]^{\frac{1}{p^*-2}} - S^{\frac{-p^*}{2}} \| |x|^{-\sigma/q} f(x) |_{q/(q-1)},$$

where

$$\int_\Omega (|x|^{-\sigma/q} f(x))^{\frac{q-1}{q}} \leqslant \mu^{\frac{1}{p^*-2}} \left[\frac{1}{2} p^* S^{\frac{p^*}{2}} \right]^{\frac{1}{p^*-2}} (1-\mu) \frac{1}{2} S^{\frac{p^*}{2}}.$$

Theorem 1 Suppose that $p^* > p$ and $-N < \beta \leqslant 0$. Then, for every function $|x|^{-\sigma/q} f(x) \in L^{q/(q-1)}(\Omega)$ and $f(x) \geqslant 0$, there exists a real number $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon \leqslant \varepsilon_0$, problem (1) has at least two positive solutions $\underline{u}$ and u in $H_{0,\beta}^{1,p}(\Omega)$.

Throughout this paper, let $H_{0,\beta}^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|$. We denote the norm of u in $H_{0,\beta}^{1,p}(\Omega)$ and $L^p(\Omega)$ by $\|u\| = (\int_\Omega |x|^{\beta} |\nabla u|^2)^{\frac{1}{2}}$, $\|u\|_{\alpha, p^*} = (\int_\Omega |x|^{\alpha} |u|^{p^*})^{\frac{1}{p^*}}$

and $|u|_p = \left(\int_\Omega |u|^p\right)^{1/p}$ respectively.

Definition 1 A sequence $\{u_m\} \in H_{0,\beta}^{1,p}(\Omega)$ is called a $(\mathrm{PS})_C$ sequence, if $I(u_m) \to c$ and $I'(u_m) \to 0$.

1 Multiple Solutions for Elliptic Equation With Critical Sobolev-Hardy Exponent

It is well known that the nontrivial solutions of problem (1) are equivalent to the nonzero critical points of the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |x|^\beta |\nabla u|^2 - \frac{1}{p^*}\int_\Omega |x|^\alpha |u|^{p^*} - \varepsilon\int_\Omega f(x)u, \quad \forall u \in H_{0\beta}^{1p}(\Omega), \tag{2}$$

which is well defined for the parameters in the previously specified intervals.

Using the duality product, we define a weak solution of problem (1) as a critical point for the functional I, there exists a function $u \in H_{0,\beta}^{1,p}(\Omega)$ such that

$$\int_\Omega |x|^\beta \nabla u \nabla \phi = \int_\Omega |x|^\alpha u^{p^*-1}\phi + \varepsilon\int_\Omega f(x)\phi, \quad \forall \phi \in H_{0\beta}^{1p'}(\Omega). \tag{3}$$

Lemma 1 If $\rho > 0$ satisfies

$$\frac{1}{2} - \frac{1}{p^*}S^{\frac{-p^*}{2}}\rho^{p^*-2} > 0, \quad 0 < \varepsilon < \rho\left[\frac{1}{2} - \frac{1}{p^*}S^{\frac{-p^*}{2}}\rho^{p^*-2}\right] - S^{\frac{-p^*}{2}} \left|\,|x|^{-\alpha/p^*} f(x)\right|_{p^*/(p^*-1)},$$

then exists positive constants τ, ρ such that

$$I(u) \geqslant \tau > 0 \text{ for } \|u\| = \rho, \forall u \in H_{0,\beta}^{1,p}(\Omega).$$

Proof By the Sobolev-Hardy inequality we know

$$\left(\int_\Omega |x|^\alpha |u|^{p^*}\right)^{\frac{1}{p^*}} \leqslant S^{\frac{-p^*}{2}}\left(\int_\Omega |x|^\beta |\nabla u|^2\right)^{\frac{1}{2}}, \quad \forall u \in H_{0,\beta}^{1,p}(\Omega).$$

Using Holder inequality and

$$\left|\int_\Omega f(x)u\right| = \int_\Omega f(x)\ |x|^{\frac{-\sigma}{q}} u\ |x|^{\frac{\sigma}{q}},$$

we can deduce

$$\left|\int_\Omega f(x)u\right| \leqslant \left(\int_\Omega |x|^\sigma |u|^q\right)^{\frac{1}{q}}\left(\int_\Omega (|x|^{-\sigma/q}f(x))^{\frac{q}{q-1}}\right)^{\frac{q}{q-1}} \leqslant$$
$$S^{\frac{-p^*}{2}}\left(\int_\Omega |x|^\beta |\nabla u|^2\right)^{\frac{1}{2}}\left(\int_\Omega (|x|^{-\sigma/q}f(x))^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}.$$

It follows from the assumptions that

$$I(u) \geqslant \left\{\frac{1}{2} - \frac{1}{p^*}S^{\frac{-p^*}{2}}\rho^{p^*-2}\right\}\rho^2 - \left(\int_\Omega (|x|^{-\sigma/q}f(x))^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}S^{\frac{-p^*}{2}} =$$
$$\rho\left\{\frac{1}{2} - \frac{1}{p^*}S^{\frac{-p^*}{2}}\rho^{p^*-2}\right\}\rho - \left(\int_\Omega (|x|^{-\sigma/q}f(x))^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}S^{\frac{-p^*}{2}}.$$

Thus

$$\frac{1}{2} - \frac{1}{p^*}S^{\frac{-p^*}{2}}\rho^{p^*-2} > 0,$$

we deduce

$$\rho = \mu^{\frac{1}{p^*-2}}\left[\frac{1}{2}p^* S^{\frac{p^*}{2}}\right]^{\frac{1}{p^*-2}}, \tag{4}$$

where $0 < \mu < 1$. From (4) and

$$\left(\int_\Omega (|x|^{-\sigma/q}f(x))^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \leqslant \rho\left\{\frac{1}{2} - \frac{1}{p^*}S^{\frac{-p^*}{2}}\rho^{p^*-2}\right\}S^{\frac{p^*}{2}}, \tag{5}$$

we can conclude that

$$\left(\int_\Omega (|x|^{-\sigma/q}f(x))^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \leqslant \mu^{\frac{1}{p^*-2}}\left[\frac{1}{2}p^* S^{\frac{p^*}{2}}\right]^{\frac{1}{p^*-2}}(1-\mu)\frac{1}{2}S^{\frac{p^*}{2}}. \tag{6}$$

Lemma 2 Let $\varepsilon\in(0,\varepsilon_0)$, then the minimal problem $\inf\{I(u):\|u\|\leqslant\rho_0\}$ is attained at weak solution $\underline{u}$ with $I(\underline{u})<0$, which satisfies $\|\underline{u}\|<\rho_0=\left(\frac{p^*}{2}S^{\frac{p^*}{2}}\right)^{\frac{1}{p^*-2}}$.

Proof Firstly we prove that $\inf\{I(u):\|u\|\leqslant\rho_0\}$ is attained. Let $\{u_m\}$ be a minimizing sequence. Hence u_m is bounded in $H^{1,p}_{0,\beta}(\Omega)$, there exist some $\underline{u}\in H^{1,p}_{0,\beta}(\Omega)$ with $\|\underline{u}\|\leqslant\rho_0$ such that

$$u_m\rightharpoonup\underline{u}\ \text{ in } H^{1,p}_{0,\beta}(\Omega),\quad u_m\to\underline{u}\ \text{ a.e. in } \Omega.$$

For $\int_\Omega f(x)u_m\to\int_\Omega f(x)\underline{u}$, refer to [4] we have

$$\begin{aligned}I(u_m)=&\ I(\underline{u})+\frac{1}{2}\int_\Omega|x|^\beta(|\nabla u_m|^2-|\nabla\underline{u}|^2)-\frac{1}{p^*}\int_\Omega|x|^\alpha(|u_m|^{p^*}-|\underline{u}|^{p^*})+o(1)\geqslant\\ &I(\underline{u})+\left(\frac{1}{2}-\frac{1}{p^*}S^{\frac{-p^*}{2}}\|u_m-\underline{u}\|^{p^*-2}\right)\|u_m-\underline{u}\|^2+o(1)\geqslant\\ &I(\underline{u})+\left(\frac{1}{2}-\frac{1}{p^*}S^{\frac{-p^*}{2}}(\rho_0)^{p^*-2}\right)\|u_m-\underline{u}\|^2+o(1)\geqslant I(\underline{u})+o(1).\end{aligned}$$

If $\rho_0=\left(\frac{p^*}{2}S^{\frac{p^*}{2}}\right)^{\frac{1}{p^*-2}}$, $I(u_m)\geqslant I(\underline{u})+o(1)$, therefore $\underline{u}$ considers $\inf\{I(u)\,\big|\,\|u\|\leqslant\rho_0\}$.

By Lemma 1, we get

$$\begin{aligned}\varepsilon<&\left\{\frac{1}{2}-\frac{1}{p^*}S^{\frac{-p^*}{2}}\left[\frac{1}{2}\left(\frac{p^*}{2}S^{\frac{p^*}{2}}\right)^{\frac{1}{p^*-2}}\right]^{p^*-2}\right\}\left[\left(\frac{p^*}{2}S^{\frac{p^*}{2}}\right)^{\frac{1}{p^*-2}}\right]-S^{\frac{-p^*}{2}}\,\left||x|^{-\sigma/q}f(x)\right|_{q(q-1)}=\\ &\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^{p^*-2}\right]\left[\frac{p^*}{2}S^{\frac{p^*}{2}}\right]^{\frac{1}{p^*-2}}-S^{\frac{-p^*}{2}}\,\left||x|^{-\sigma/q}f(x)\right|_{q/(q-1)}=\varepsilon_0.\end{aligned}$$

Hence we have $I(u)\geqslant\tau>0$, $\forall\|u\|=\rho_0$, but $I(0)=0$, so that

$$I(\underline{u})\leqslant I(0)=0$$

and $\|\underline{u}\|<\rho_0$.

Let

$$\Gamma=\{\gamma(t)\in C^1([0,1],H^{1,p}_{0,\beta}(\Omega)):\gamma(0)=\underline{u},\ \gamma(1)=e\},$$

where $e\in H^{1,p}_{0,\beta}(\Omega)$ satisfies $I(e)<0$.

Let $c=\inf\limits_{\gamma\in\Gamma}\sup\limits_{t\in[0,1]}I(\gamma(t))$, by Lemma 1, when $\varepsilon\in(0,\varepsilon_0)$, $c>0$. By Mountain Pass Lemma without (PS) condition[2], there is $u_m\in H^{1,p}_{0,\beta}(\Omega)$, such that

$$I(u_m)\to c,\quad I'(u_m)\to 0 \tag{7}$$

From above we easily prove that $\{u_m\}$ is bounded in $H^{1,p}_{0,\beta}(\Omega)$, by extracting subsequence, still denoted by $\{u_m\}$, then there is a $u\in H^{1,p}_{0,\beta}(\Omega)$ such that

$$u_m\rightharpoonup u\ \text{ in } H^{1,p}_{0,\beta}(\Omega),\quad u_m\to u\ \text{ a.e. in } \Omega. \tag{8}$$

By (7) we know that u satisfies (1).

Lemma 3 If $c<I(\underline{u})+\left(\frac{1}{2}-\frac{1}{p^*}\right)S^{p^*/(p^*-2)}$, then $u\neq\underline{u}$.

Proof We prove by contradiction. Suppose that $u=\underline{u}$. Then we have

$$c+o(1)=I(u_m)=I(\underline{u})+\frac{1}{2}\int_\Omega|x|^\beta(|\nabla u_m|^2-|\nabla\underline{u}|^2)-\frac{1}{p^*}\int_\Omega|x|^\alpha(|u_m|^{p^*}-|\underline{u}|^{p^*})+o(1), \tag{9}$$

$$\begin{aligned}o(1)=\langle I'(u_m),u_m\rangle=&\int_\Omega|x|^\beta|\nabla u_m|^2-\int_\Omega|x|^\alpha|u_m|^{p^*}-\int_\Omega f(x)u_m=\\ &\int_\Omega|x|^\beta|\nabla\underline{u}|^2-\int_\Omega|x|^\alpha|\underline{u}|^{p^*}-\int_\Omega f(x)\underline{u}+\\ &\int_\Omega|x|^\beta(|\nabla u_m|^2-|\nabla\underline{u}|^2)-\int_\Omega|x|^\alpha(|u_m|^{p^*}-|\underline{u}|^{p^*})+o(1)=\end{aligned}$$

$$\int_\Omega |x|^\beta(|\nabla u_m|^2-|\nabla u|^2)-\int_\Omega |x|^\alpha(|u_m|^{p^*}-|u|^{p^*})+o(1). \tag{10}$$

If there exists a subsequence (still denoted by u_m), such that

$$\int_\Omega |x|^\beta(|\nabla u_m|^2-|\nabla \underline{u}|^2)\to k\geqslant 0,$$

from (10) we obtain

$$\int_\Omega |x|^\alpha(|u_m|^{p^*}-|\underline{u}|^{p^*})\to k.$$

On the other hand, by Sobolev-Hardy inequality, we have

$$\int_\Omega |x|^\beta(|\nabla u_m|^2-|\nabla \underline{u}|^2)\geqslant S\left(\int_\Omega |x|^\alpha(|u_m|^{p^*}-|\underline{u}|^{p^*})\right)^{\frac{2}{p^*}},$$

setting $m\to\infty$, we have $k\geqslant Sk^{\frac{2}{p^*}}$, so that $k=0$ or $k\geqslant S^{p^*/(p^*-2)}$.

Letting $k=0$, we get that $u_m\to \underline{u}$ strongly in $H^{1,p}_{0,\beta}(\Omega)$, which gives that

$$c+o(1)=I(u_m)\to I(u)\leqslant 0,$$

this is in contradiction with $c>0$. So that $k\geqslant S^{p^*/(p^*-2)}$. By (9) and (10), we have

$$c+o(1)=I(\underline{u})+\left(\frac{1}{2}-\frac{1}{p^*}\right)\int_\Omega |x|^\beta(|\nabla u_m|^2-|\nabla \underline{u}|^2)+o(1)\geqslant$$

$$I(\underline{u})+\left(\frac{1}{2}-\frac{1}{p^*}\right)S^{p^*/(p^*-2)}+o(1).$$

This is a contradiction.

Lemma 4[5] **(Calculus Lemma)** For every $1\leqslant p\leqslant 3$, there exists a constant c (depending on p) such that for $\alpha,\beta\in\mathbf{R}$, we have

$$||\alpha+\beta|^p-|\alpha|^p-|\beta|^p-p\alpha\beta(|\alpha|^{p-2}+|\beta|^{p-2})|\leqslant\begin{cases}c|\alpha||\beta|^{p-1}, & |\alpha|\geqslant|\beta|,\\ c|\beta||\alpha|^{p-1}, & |\alpha|\leqslant|\beta|.\end{cases}$$

For $p\geqslant 3$, there exists a constant c (depending on p) such that for $\alpha,\beta\in\mathbf{R}$, we have

$$||\alpha+\beta|^p-|\alpha|^p-|\beta|^p-p\alpha\beta(|\alpha|^{p-2}+|\beta|^{p-2})|\leqslant c(|\beta|^2|\alpha|^{p-2}+|\alpha|^2|\beta|^{p-2}).$$

From this inequality, we can actually deduce the following more convenient result for any $p\geqslant 1$:

$$||\alpha+\beta|^p-|\alpha|^p-|\beta|^p-p\alpha\beta(|\alpha|^{p-2}+|\beta|^{p-2})|\leqslant 2c(|\alpha|^{p-1}|\beta|+|\alpha||\beta|^{p-1}).$$

2 Proof of Theorem 1

It is sufficient to prove:

$$c<I(\underline{u})+\left(\frac{1}{2}-\frac{1}{p^*}\right)S^{p^*/(p^*-2)}+o(1).$$

Let ϕ be a cut-off function in $C_0^\infty(\Omega)$ defined as

$$\phi(x)=\begin{cases}1, & \text{if } x\in B_\varepsilon(0),\\ 0, & \text{if } x\in B_{2\varepsilon}(0),\end{cases}$$

and $|\nabla\phi|<\frac{2}{\varepsilon}$, $B_\varepsilon(0)$ is a small ball centered at the origin with radius ε.

Set

$$u_\varepsilon(x)=\phi(x)U_\varepsilon(x),$$

where

$$U_\varepsilon=\frac{c_\varepsilon}{(\varepsilon+|x|^{\alpha-\beta+2})^{\frac{N+\beta-2}{\alpha-\beta+2}}},\quad \varepsilon>0.$$

And define

$$v_\varepsilon(x)=\frac{u_\varepsilon(x)}{\left(\int_\Omega |x|^\alpha|u_\varepsilon(x)|^{p^*}\right)^{\frac{1}{p^*}}}.$$

By the equation (3) we have

$$\int_\Omega |x|^\beta |\nabla \underline{u}| \nabla v_\varepsilon = \int_\Omega |x|^\alpha |\underline{u}|^{p^*-2} \underline{u} v_\varepsilon + \int_\Omega f(x) v_\varepsilon .$$

Therefore

$$I(\underline{u} + t v_\varepsilon) = I(\underline{u}) + \frac{t^2}{2}\int_\Omega |x|^\beta |\nabla v_\varepsilon|^2 - \frac{t^{p^*}}{p^*}\int_\Omega |x|^\alpha |v_\varepsilon|^{p^*} -$$

$$\frac{1}{p^*}\int_\Omega |x|^\alpha (|\underline{u} + t v_\varepsilon|^{p^*} - \underline{u}^{p^*} - |t v_\varepsilon|^{p^*} - p^* t \underline{u}^{p^*-1} v_\varepsilon).$$

By simple calculation we obtain

$$\int_\Omega |x|^\beta |\nabla u_\varepsilon|^2 = \int_{\mathbb{R}^N} |x|^\beta |\nabla U_\varepsilon|^2 + O(1) = \varepsilon^{\frac{-(2-N-\beta)}{\alpha-\beta+2}} \int_{\mathbb{R}^N} |x|^\beta |\nabla U_1|^2 + O(1),$$

$$\int_\Omega |x|^\alpha |\nabla u_\varepsilon|^{p^*} = \int_{\mathbb{R}^N} |x|^\alpha |U_\varepsilon|^{p^*} + O(1) = \varepsilon^{\frac{-(N+\alpha)}{\alpha-\beta+2}} \int_{\mathbb{R}^N} |x|^\alpha |U_1|^{p^*} + O(1),$$

$$\int_\Omega |x|^\alpha |u_\varepsilon|^q = \varepsilon^{\frac{\alpha-(N+\beta-2)q+N}{\alpha-\beta+2}} \int_{\mathbb{R}^N} |x|^\alpha |U_1|^q + O(1),$$

$$\int_\Omega |x|^\beta |\nabla v_\varepsilon|^2 = \frac{\int_{\mathbb{R}^N} |x|^\beta |\nabla U_1|^2 + O\left(\varepsilon^{\frac{N+\beta-2}{\alpha-\beta+2}}\right)}{\left[\int_{\mathbb{R}^N} |x|^\alpha |U_1|^{p^*} + O\left(\varepsilon^{\frac{N+\alpha}{\alpha-\beta+2}}\right)\right]^{2/p^*}} = S + O\left(\varepsilon^{\frac{N+\beta-2}{\alpha-\beta+2}}\right),$$

$$\int_\Omega |x|^\alpha |v_\varepsilon|^{p^*} = 1,$$

$$\int_\Omega |x|^\alpha |v_\varepsilon|^q = k_1 \varepsilon^{\frac{\alpha-(N+\beta-2)q/2+N}{\alpha-\beta+2}},$$

where $k_1 \geqslant k > 0$, k is a constant.

For $t_\varepsilon \geqslant 0$, $\max_{t\geqslant 0} I(\underline{u} + t v_\varepsilon)$ is attained, we easily verify that $t_\varepsilon \geqslant t_0 > 0$. Let $t_\varepsilon > 0$, such that

$$\max_{t\geqslant 0} I(\underline{u} + t v_\varepsilon) = I(\underline{u} + t_\varepsilon v_\varepsilon),$$

so that

$$\max_{t\geqslant 0} I(\underline{u} + t v_\varepsilon) = I(\underline{u}) + \frac{t_\varepsilon^2}{2} S - \frac{1}{p^*} t_\varepsilon^{p^*} + O\left(\varepsilon^{\frac{N+\beta-2}{\alpha-\beta+2}}\right) - \int_\Omega |x|^\alpha \underline{u} (t_\varepsilon v_\varepsilon)^{p^*-1} -$$

$$\frac{1}{p^*}\int_\Omega |x|^\alpha [|\underline{u} + t_\varepsilon v_\varepsilon|^{p^*} - \underline{u}^{p^*} - (t_\varepsilon v_\varepsilon)^{p^*}] + \int_\Omega [\underline{u}^{p^*-1}(t_\varepsilon v_\varepsilon) + \underline{u}(t_\varepsilon v_\varepsilon)^{p^*-1}]. \tag{11}$$

Because $\underline{u} \geqslant \tau$, for $x \in B_\varepsilon$, and $\underline{u}$ is bounded in Ω, so

$$\int_\Omega |x|^\alpha \underline{u} |u_\varepsilon|^{p^*-1} \geqslant \tau \int_{\mathbb{R}^N} |x|^\alpha |U_\varepsilon|^{p^*-1} + O(1) =$$

$$\tau \varepsilon^{\frac{\alpha-(N+\beta-2)(p^*-1)+N}{\alpha-\beta+2}} \int_{\mathbb{R}^N} |x|^\alpha |U_1|^{p^*-1} + O(1), \tag{12}$$

and thus

$$\int_\Omega |x|^\alpha \underline{u} v_\varepsilon^{p^*-1} \geqslant \tau k' \varepsilon^{(\alpha-(N+\beta-2)\frac{p^*-1}{2}+N)/(\alpha-\beta+2)}. \tag{13}$$

Let

$$I_\varepsilon = \int_\Omega |x|^\alpha [|\underline{u} + t_\varepsilon v_\varepsilon|^{p^*} - \underline{u}^{p^*} - (t_\varepsilon v_\varepsilon)^{p^*} - p^* \underline{u}^{p^*-1}(t_\varepsilon v_\varepsilon) - p^* \underline{u}(t_\varepsilon v_\varepsilon)^{p^*-1}],$$

we argue in two cases $p^* > 3$ and $2 < p^* \leqslant 3$. Using the hypotheses of Lemma 4 when $p^* > 3$, by

$$|(1+a)^{p^*} - 1 - a^{p^*} - p^* a - p^* a^{p^*-1}| \leqslant c a^2, \quad \forall\ |a| \leqslant 1,$$

we have

$$|I_\varepsilon| \leqslant c\int_{\{\underline{u} > t_\varepsilon v_\varepsilon\}} |x|^\alpha \underline{u}^{p^*-2}(t_\varepsilon v_\varepsilon)^2 + c\int_{\{\underline{u} \leqslant t_\varepsilon v_\varepsilon\}} |x|^\alpha \underline{u}^2 (t_\varepsilon v_\varepsilon)^{p^*-2} \leqslant$$

$$c\int_\Omega |x|^\alpha v_\varepsilon^2 + c\int_\Omega |x|^\alpha \underline{u}^{1+\gamma} (t_\varepsilon v_\varepsilon)^{p^*-1-\gamma} \leqslant$$

$$c\int_\Omega |x|^\alpha v_\varepsilon^2 + c\int_\Omega |x|^\alpha v_\varepsilon^{p^*-1-\gamma} \ (0<\gamma<1).$$

Since $p^*-1>2$, we have

$$|I_\varepsilon| = o\left(\varepsilon^{(\alpha-(N+\beta-2)\frac{p^*-1}{2}+N)/(\alpha-\beta+2)}\right). \tag{14}$$

When $2<p^*\leqslant 3$, by

$$||1+a|^{p^*}-1-a^{p^*}-p^*a-p^*a^{p^*-1}|\leqslant c|a|^2\leqslant c|a|^{p^*-1},\quad \forall\ |a|\leqslant 1,$$

we have

$$|I_\varepsilon|\leqslant c\int_{\{\underline{u}>t_\varepsilon v_\varepsilon\}}|x|^\alpha \underline{u}\,|t_\varepsilon v_\varepsilon|^{p^*-1} + c\int_{\{\underline{u}\leqslant t_\varepsilon v_\varepsilon\}}|x|^\alpha \underline{u}^{p^*-1}(t_\varepsilon v_\varepsilon)\leqslant$$

$$c\int_{\{\underline{u}>t_\varepsilon v_\varepsilon\}}|x|^\alpha \underline{u}^{1+\gamma}|t_\varepsilon v_\varepsilon|^{p^*-1-\gamma} + c\int_{\{\underline{u}\leqslant t_\varepsilon v_\varepsilon\}}|x|^\alpha \underline{u}^{1+\gamma}(t_\varepsilon v_\varepsilon)^{p^*-1-\gamma}\leqslant$$

$$c\int_\Omega |x|^\alpha v^{p^*-1-\gamma} = o\left(\varepsilon^{(\alpha-(N+\beta-2)\frac{p^*-1}{2}+N)/(\alpha-\beta+2)}\right), \tag{15}$$

where $0<\gamma<1$. Combining (11) ~ (15) we have

$$\max_{t\geqslant 0} I(\underline{u}+tv_\varepsilon)\leqslant I(\underline{u})+\frac{t_\varepsilon^2}{2}S-\frac{1}{p^*}t_\varepsilon^{p^*}-t_0^{p^*-1}\tau' k'\varepsilon^{(\alpha-(N+\beta-2)\frac{p^*-1}{2}+N)/(\alpha-\beta+2)}+$$

$$o\left(\varepsilon^{(\alpha-(N+\beta-2)\frac{p^*-1}{2}+N)/(\alpha-\beta+2)}\right)+O\left(\varepsilon^{(\alpha+\beta-2)/(\alpha-\beta+2)}\right). \tag{16}$$

Because

$$\frac{t_\varepsilon^2}{2}S-\frac{1}{p^*}t_\varepsilon^{p^*}\leqslant\max_{t\geqslant 0}\left(\frac{t^2}{2}S-\frac{1}{p^*}t^{p^*}\right)=\left(\frac{1}{2}-\frac{1}{p^*}\right)S^{p^*/(p^*-2)},$$

by (16), for $\varepsilon>0$ sufficiently small, we have

$$\max_{t\geqslant 0} I(\underline{u}+tv_\varepsilon)<\left(\frac{1}{2}-\frac{1}{p^*}\right)S^{p^*/(p^*-2)}+I(\underline{u}).$$

This completes the proof of Theorem 1.

3 Conclusions

In this work we present the multiple solutions for elliptic equation with critical Sobolev – Hardy exponent, then we establish some results of lemmas using Sobolev Hardy inequality and Mountain Pass Lemma without the Palais – Smale condition which are needed for the proof of the Theorem 1, and then we find the problem (1) has at least two positive solutions $\underline{u}$ and u in $H^{1,p}_{0,\beta}(\Omega)$.

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