Some New Common Fixed Point Theorems for Weakly Compatible Mappings

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Abstract Some new common fixed point theorems are established under strict contractive conditions for weakly compatible mappings satisfying the property (E. A). The main result of Aamri and Moutawakil is improved and generalized.

Key words common fixed point, the property (E. A), weakly compatible

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几个新的弱相容映射的公共不动点定理

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[摘要] 在严格压缩条件下, 对满足(E. A)性质的弱相容映射建立几个新的的公共不动点定理. 改进并推广了Aamri和 Moutawakil的一个主要结果.

[关键词] 公共不动点, (E. A)性质, 弱相容

In 1986, Jungck[1] introduced the concept of compatible mapping and proved some common fixed point theorems of compatible mappings in metric space. However the study of common fixed points of noncompatible mapping is also very interesting[2-5]. In 2001, Aamri and Moutawakil[6] defined a new property for a pair of mappings, i. e., so-called the property (E. A), which generalize the concept of noncompatible mappings. By using the property (E. A), they obtained the following common fixed point theorem:

Theorem A Let A, B, T and S be self mappings of a metric space (X, d), and let F: $\mathbf{R}^+ \to \mathbf{R}^+$ be a non-decreasing function satisfying $0 < F(t) < t$ for each $t > 0$ such that

(1)* $d(Ax, By) \leqslant F(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$, $\forall\ x, y \in X$;

(2) (A, S) and (B, T) are weakly compatible;

(3) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

(4)* (A, S) or (B, T) satisfies the property (E. A).

If the range of the one of the mappings A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

We find that the author's proof is half-baked. In the original proof, the authors only examined the case of that (B, T) satisfies the property (E. A). They consider that when (A, S) satisfies the property (E. A), the approach is similar. Actually, not so, since the condition (1) is not symmetrical with respect to (A, S) and (B, T). We find that if (A, S) satisfies the property (E. A) (and (B, T) does not satisfy that), then the existing

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conditions seem not to be able to assure that the conclusion of the theorem holds still. In this paper, we give some new common fixed point theorems under strict contractive conditions for mappings satisfying the property (E.A), which improve and generalize the corresponding result of Aamri and Moutawakil in [6].

1 Some Basic Concepts

For the convenience of the reading we recall some basic concepts which will be needed in the sequel.

Definition 1[1,7] Let S and T be two self-maps of a metric space (X, d).

(1) S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$.

(2) S and T are said to be weakly compatible if they commute at their coincidence points, i.e., if $Tu = Su$ for some $u\in X$, then $TSu = STu$.

Definition 2[6] Let S and T be two selfmappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t\in X$.

2 Main Results

Theorem 1 Let A, B, T and S be selfmappings of a metric space (X, d). If $B(X)$ or $S(X)$ is a closed subspace of X, and the following conditions are satisfied:

(1) $d(Ax, By) < \max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\}$, $\forall\ x, y\in X$ with $Ax\neq By$;

(2) (A, S) and (B, T) are weakly compatible;

(3) $A(X)\subset T(X)$ and $B(X)\subset S(X)$;

(4) (B, T) satisfies the property (E.A).

Then A, B, S and T have a unique common fixed point.

Proof Since (B, T) satisfies the property (E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\in X$. Note that $B(X)\subset S(X)$. Hence there exists a sequence (y_n) in X such that $Bx_n = Sy_n$, and so $\lim_{n\to\infty} Sy_n = t$.

Suppose $S(X)$ is a closed subspace of X. Note that $\{Sy_n\}\subset S(X)$ and $Sy_n\to t$. Hence $t = Su$ for some $u\in X$, and so $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su$. It is not difficult to prove that $Su = Au$. We consider the following two cases:

If there exists $N\in \mathbf{N}$ such that $Au\neq Bx_n$ for all $n\geqslant N$, then it follows from (1) that

$$d(Au, Bx_n) < \max\{d(Su, Tx_n), d(Su, Bx_n), d(Tx_n, Bx_n)\}.$$

Letting $n\to\infty$, it follows from the above inequality that $d(Au, Su)\leqslant d(Su, Su) = 0$. Hence $Su = Au$.

If there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $Au = Bx_{n_i}$, $i = 1, 2, \cdots, n$, then it is clear that $Au = Su$.

Since A and S are weakly compatible and $Au = Su$, we have $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.

On the other hand, since $A(X)\subset T(X)$, there exists $v\in X$ such that $Tv = Au$. We can claim that $Tv = Bv$. If not, then by (1), we get

$$d(Au, Bv) < \max\{d(Su, Tv), d(Su, Bv), d(Tv, Bv)\} = \max\{0, d(Au, Bv), d(Au, Bv)\} = d(Au, Bv),$$

which is a contradiction. Hence $Au = Bv$, and so $Tv = Bv$. The weak compatibility of B and T implies that $TTv = TBv = BTv = BBv$.

Let us show that Au is a common fixed point of A, B, S and T. If $AAu\neq Au$, i.e., $AAu\neq Bv$, it follows from (1) that

$$d(Au, AAu) = d(AAu, Bv) < \max\{d(SAu, Tv), d(SAu, Bv), d(Tv, Bv)\} =$$
$$\max\{d(AAu, Au), d(AAu, Au), 0\} = d(AAu, Au),$$

which is a contradiction, and hence $Au = AAu$. Therefore $Au = AAu = SAu$. This shows that Au is a common fixed point of A and S. Note that $d(Bv, BBv) = d(Au, BBv)$. Similarly, by using (1), it is easy to prove that Bv is a common fixed point of B and T. Since $Au = Bu$, we conclude that Au is a common fixed point of A, B, S and T.

Suppose $B(X)$ is a closed subspace of X. Note that $\{Bx_n\} \subset B(X)$ and $Bx_n \to t$. Hence $t = Bb$ for some $b \in X$. Since $B(X) \subset S(X)$, there exists $u \in X$ such that $Bb = Su$, and so we have

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su.$$

The subsequent proof is similar to the case in which $T(X)$ is assumed to be a closed subspace of X.

Finally, we prove the uniqueness. Suppose that $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$. If $u \neq v$, i.e., $Au \neq Bv$, by (1) we get

$$d(u, v) = d(Au, Bv) < \max\{d(Su, Tv), d(Su, Bv), d(Tv, Bv)\} = \max\{d(Au, Bv)\} = d(u, v),$$

which is a contradiction. Therefore $u = v$.

Remark It is evident that the condition $(1)^*$ of Theorem A implies the condition (1) of Theorem 1, and every complete subspace of X is closed. Hence Theorem 1 is a generalization of Theorem A when only assuming that (B, T) satisfies the property (E. A).

If A and B, S and T are interchanged in Theorem 1 respectively, we get the following theorem.

Theorem 2 Let A, B, T and S be selfmappings of a metric space (X, d). If $A(X)$ or $T(X)$ is a closed subspace of X, and the following conditions are satisfied:

$(1)'$ $d(Ax, By) < \max\{d(Sx, Ty), d(Ax, Ty), d(Ax, Sx)\}$, $\forall\ x, y \in X$ with $Ax \neq By$;

(2) (A, S) and (B, T) are weakly compatible;

(3) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

$(4)'$ (A, S) satisfies the property (E. A).

Then A, B, S and T have a unique common fixed point.

Let $F: \mathbf{R}^+ \to \mathbf{R}^+$ be a function. We denote the following properties by (F-1) and (F-2) respectively:

(F-1) F is nondecreasing and upper semi-continuous on $\mathbf{R}^+$,

(F-2) $0 < F(t) < t$ for each $t > 0$.

The following theorem is a correction of Theorem A when we suppose that (A, S) satisfies the property (E. A).

Theorem 3 Let A, B, T and S be selfmappings of a metric space (X, d), $A(X)$ or $T(X)$ be the closed subspace of X, and let $F: \mathbf{R}^+ \to \mathbf{R}^+$ be a function satisfying (F-1) and (F-2). Suppose that the following conditions are satisfied:

$(1)^*$ $d(Ax, By) \leqslant F(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$, $\forall\ x, y \in X$;

(2) (A, S) and (B, T) are weakly compatible;

(3) $A(X) \subset T(X)$ and $B(X) \subset S(X)$;

$(4)'$ (A, S) satisfies the property (E. A).

Then A, B, S and T have a unique common fixed point.

Proof Since (A, S) satisfies the property (E. A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Note that $A(X) \subset T(X)$. Hence there exists in X a sequence (y_n) such that $Ax_n = Ty_n$, and so $\lim_{n\to\infty} Ty_n = t$.

Suppose $T(X)$ is a closed subspace of X. Note that $\{Ty_n\} \subset T(X)$ and $Ty_n \to t$. Hence $t = Tu$ for some $u \in X$. So, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = Tu.$$

From $(1)^*$ we have

$$d(Ax_n, Bu) \leqslant F(\max\{d(Sx_n, Tu), d(Sx_n, Bu), d(Tu, Bu)\}).$$

Since F is upper semi-continuous, it follows

$$d(Tu, Bu) \leqslant \overline{\lim_{n\to\infty}} F(\max\{d(Sx_n, Tu), d(Sx_n, Bu), d(Tu, Bu)\}) \leqslant F(d(Tu, Bu)),$$

which implies that $Tu = Bu$ by (F-2). The weak compatibility of B and T implies that $TTu = TBu = BTu =$

BBu.

On the other hand, since $B(X)\subset S(X)$, there exists $v\in X$ such that $Sv=Bu$. We can claim that $Sv=Av$. In fact, using $(1)^*$, we have

$$d(Av,Bu)\leqslant F(\max\{d(Sv,Tu),d(Sv,Bu),d(Tu,Bu)\})=$$
$$F(\max\{d(Bu,Bu),d(Bu,Bu),d(Bu,Bu)\})=0,$$

which implies that $Av=Bu$, and so $Av=Sv$. The weak compatibility of A and S implies that $AAv=ASv=SAv=SSv$.

Let us show that Av is a common fixed point of A, B, S and T. In view of $(1)^*$, it follows

$$d(Av,AAv)=d(AAv,Bu)\leqslant F(\max\{d(SAv,Tu),d(SAv,Bu),d(Tu,Bu)\})=$$
$$F(\max\{d(AAv,Av),d(AAv,Av),0\})=F(d(AAv,Av)),$$

which implies that $Av=AAv=SAv$ by (F-2). Therefore, Av is a common fixed point of A and S. Similarly, it is not difficult to prove that Bu is a common fixed point of B and T. Since $Av=Bu$, we conclude that Av is a common fixed point of A, B, S and T.

Suppose $A(X)$ is a closed subspace of X. Note that $\{Ax_n\}\subset A(X)$ and $Ax_n\to t$. Hence $t=Ab$ for some $b\in X$. Since $A(X)\subset T(X)$, there exists $u\in X$ such that $Ab=Tu$, and so we have

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}Ty_n=Tu.$$

The subsequent proof is similar to the case in which $T(X)$ is assumed to be a closed subspace of X.

By (F-2), it is easy to see that the condition $(1)^*$ implies the condition (1) in Theorem 1. Hence, the proof of uniqueness is similar to that of uniqueness in Theorem 1.

As a corollary of Theorem 1 and Theorem 3, we obtain the following result which is a correction of Theorem A.

Corollary 1 Let A, B, T and S be selfmappings of a metric space (X,d), and let $F:\mathbf{R}^+\to\mathbf{R}^+$ be a function satisfying the conditions (F-1) and (F-2) such that

$(1)^*$ $d(Ax,By)\leqslant F(\max\{d(Sx,Ty),d(Sx,By),d(Ty,By)\})$, $\forall\ x,y\in X$;

(2) (A,S) and (B,T) are weakly compatible;

(3) $A(X)\subset T(X)$ and $B(X)\subset S(X)$;

$(4)^*$ (A,S) or (B,T) satisfies the property (E.A).

If the range of the one of the mappings A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

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