A Mortar Element Method for Rotated Q_1 Element

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Abstract A mortar element version for rotated Q_1 element is proposed. The mortar condition is only dependent on the degrees of the freedom on subdomains interfaces; the optimal error estimate is obtained for the rotated Q_1 mortar element method.

Key words mortar finite element, mortar condition, rotated Q_1 element

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一种 Mortar型旋转 Q_1元方法

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[摘要] 提出了一种 mortar型旋转 Q_1 元方法, 相应的 mortar条件仅依赖于子区域边界上的自由度; 并得到了较优的误差估计.

[关键词] mortar有限元, mortar条件, 旋转 Q_1元

The mortar element method is a nonconforming domain decomposition method with non-overlapping subdomains. The meshes on different subdomains need not align across subdomains interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. Recently, the method has been studied extensively and many results have been obtained[1, 2]. Marcinkowski considered the mortar element method with locally P_1 nonconforming elements and obtained the optimal error estimate in [3], but the mortar condition is dependent on the degrees of the freedom on the interfaces and those near the interfaces. In [4], Bi and Li also considered the mortar element method with locally P_1 nonconforming element, and the mortar condition is only dependent on the degrees of the freedom on the interfaces.

In [2], Chen and Xu proposed the mortar element for rotated Q_1 element, similar to [3], the mortar condition is relative to the degrees of the freedom on the interfaces and those near the interfaces. In this paper, based on [4], we also consider the mortar element for rotated Q_1 element, and the mortar condition is only correlate with the degrees of the freedom on the interfaces. By virtue of a local map on the interfaces, we construct mortar condition across the interfaces, and the optimal error estimate for rotated Q_1 mortar element method is proved.

The remainder of this paper is organized as follows. In section 1 we introduce model problem, the rotated Q_1 mortar element method and some notations. Some technical lemmas are given in section 2. Section 3 proves the optimal error estimate. Last section gives numerical experiments showing the optimality of our theoretical results. For convenience, the symbol $\preccurlyeq$, $\succcurlyeq$, and $\simeq$ will be used in this paper, and $x_1 \preccurlyeq y_1$, $x_2 \succcurlyeq y_2$, and $x_3 \simeq y_3$ mean that $x_1 \leqslant C_1 y_1$, $x_2 \geqslant C_2 y_2$, and $C_3 x_3 \leqslant y_3 \leqslant C_4 x_3$ for some constants C_1, C_2, C_3, and C_4 that are independent of

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mesh size.

1 Model Problem

Let $\Omega \subset \mathbf{R}^2$ be a rectangular or $L-$ shape bounded domain with boundary $\partial\Omega$. Consider the following model problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

the variational formulation of (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad (f, v) = \int_\Omega f v \,\mathrm{d}x. \tag{3}$$

Partition Ω into geometrically conforming rectangular substructures, i. e.,

$$\Omega = \bigcup_{k=1}^{N} \Omega_k \text{ and } \Omega_k \cap \Omega_l = \varnothing, \ k \neq l,$$

$\Omega_k \cap \Omega_l$ is empty set or a vertex or an edge for $k \neq l$. With each Ω_k we associate a quasi-uniform triangulation $\mathcal{T}_h(\Omega_k)$ made of elements that are rectangles whose edges are parallel to $x-$ axis or $y-$ axis. The mesh parameter h_k is the diameter of the largest element in $\mathcal{T}_h(\Omega_k)$. Let Γ_{kl} denote the open edge that is common to Ω_k and Ω_l, the interface $\Gamma = \bigcup \partial\Omega_k \setminus \partial\Omega$ is broken into a set of straight segments Γ_{kl}. Let $\Omega_{k,h}$ and $\partial\Omega_{k,h}$ be the sets of vertices of the triangulation $\mathcal{T}_h(\Omega_k)$ that are in Ω_k and $\partial\Omega_k$ respectively.

We construct the rotated Q_1 element for each triangulation $\mathcal{T}_h(\Omega_k)$ as follows:

$$X_h(\Omega_k) = \{v \in L^2(\Omega_k) \mid v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 (x^2 - y^2),$$

$$a_E^i \in \mathcal{R}, \ \int_{\partial E \cap \partial\Omega} v|_{\partial\Omega} \,\mathrm{d}s = 0, \ \forall E \in \mathcal{T}_h(\Omega_k);$$

$$\text{for } E_1, E_2 \in \mathcal{T}_h(\Omega_k), \ \text{ if } \partial E_1 \cap \partial E_2 = e, \text{ then } \int_e v|_{\partial E_1} \mathrm{d}s = \int_e v|_{\partial E_2} \mathrm{d}s\},$$

with norm and seminorm

$$\|v\|_{H_h^1(\Omega_k)} = \Big(\sum_{E \in \mathcal{T}_h(\Omega_k)} \|v\|_{H^1(E)}^2\Big)^{\frac12}, \quad |v|_{H_h^1(\Omega_k)} = \Big(\sum_{E \in \mathcal{T}_h(\Omega_k)} |v|_{H^1(E)}^2\Big)^{\frac12}.$$

The global discrete space is defined by

$$X_h(\Omega) = \prod_{k=1}^{N} X_h(\Omega_k),$$

with norm $\|v\|_{H_h^1(\Omega)} = \Big(\sum_{k=1}^{N} \|v\|_{H_h^1(\Omega_k)}^2\Big)^{\frac12}$ and seminorm $|v|_{H_h^1(\Omega)} = \Big(\sum_{k=1}^{N} |v|_{H_h^1(\Omega_k)}^2\Big)^{\frac12}$.

Since Γ_{kl} inherits two different triangulations, we denote one of the sides of Γ_{kl} as mortar by $\gamma_{m(k)}$ and the other as nonmortar by $\delta_{m(l)}$, then $\gamma_{m(k)} = \delta_{m(l)} = \Gamma_{kl}$, by $\mathcal{T}_h^k(\gamma_{m(k)})$ and $\mathcal{T}_h^l(\delta_{m(l)})$ denote the different triangulations across Γ_{kl}. (Assume the fine side is chosen as mortar, i. e., $h_k \leqslant h_l$.) Define $S_h(\delta_{m(l)})$ to be a subspace of $L^2(\Gamma_{kl})$, such that its functions are piecewise constants on $\mathcal{T}_h^l(\delta_{m(l)})$. The dimension of $S_h(\delta_{m(l)})$ is equal to the number of elements on the $\delta_{m(l)}$. For each nonmortar edge $\delta_{m(l)}$, define an L^2- projection operator $Q_\delta: L^2(\Gamma_{kl}) \to S_h(\delta_{m(l)})$ by

$$(Q_\delta v, \phi)_{L^2(\delta_{m(l)})} = (v, \phi)_{L^2(\delta_{m(l)})}, \quad \forall \phi \in S_h(\delta_{m(l)}). \tag{4}$$

Similarly we can define $S_h(\gamma_{m(k)})$ and Q_γ.

In the sequel, let $\mathcal{T}_{h/2}(\Omega_k)$ be the partition which is constructed by connecting midpoints of the opposite edges of elements of $\mathcal{T}_h(\Omega_k)$. Introduce an auxiliary conforming bilinear finite element space

$$V_k^{h/2} = V^{h/2}(\Omega_k) = \{v \in C^0 \mid v|_K \text{ is bilinear}, \ \forall K \in \mathcal{T}_{h/2}(\Omega_k)\}.$$

Let $V^{h/2} = \prod_{k=1}^{N} V_k^{h/2}$ and $V_k^{h/2}(s) = V_k^{h/2}|_s$ for $s \subset (\Gamma \cup \partial\Omega)$.

We introduce a local map $I^{\gamma} \to V_k^{h/2}(\gamma)$ defined as follows

(1) If P is a midpoint of $e \in \gamma$, then $I^{\gamma} v(P) = \frac{1}{|e|}\int_e v\,\mathrm{d}s$;

(2) If P is an end point of γ, $I^{\gamma} v(P) = I^{\gamma} v(P_{CR})$, where P_{CR} is the nearest midpoint of P on γ.

(3) If P is a vertex of an element $\mathcal{T}_h(\gamma)$, $I^{\gamma} v(P) = \frac{1}{2}(I^{\gamma} v(P_l) + I^{\gamma} v(P_r))$, where P_l, P_r are the left and right neighboring midpoints of P on γ.

Now, we can define the global discrete space for mortar element

$$V_h = \{v \in X_h(\Omega) \mid Q_{\delta}(I^{\gamma} Q_{\gamma} v_k) = Q_{\delta} v_l,\ \forall\ \delta_{m(l)} = \gamma_{m(k)} \subset \Gamma\},$$

where $v_k = v|_{\gamma_{m(k)}}$ and $v_l = v|_{\delta_{m(l)}}$. The condition on Γ is called mortar condition. Our discrete problem of (2) is find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h),\quad \forall\ v_h \in V_h, \tag{5}$$

where

$$a_h(u_h, v_h) = \sum_{k=1}^{N} a_{h,k}(u_h, v_h),\ a_{h,k}(u_h, v_h) = \sum_{E \in \mathcal{T}_h(\Omega_k)} (\nabla u_h, \nabla v_h)_E.$$

Obviously, the form $a_h(\cdot, \cdot)$ is positive definite, and the problem (5) has a unique solution.

2 Some Technical Lemmas

To reach our conclusion, we present some auxiliary technical lemmas.

Define an operator $\mathcal{M}_k$: $X_h(\Omega_k) \to V^{h/2}(\Omega_k)$ as Definition 3.1 in [4].

For the operator $\mathcal{M}_k$, we have the following result.

Lemma 1 For any $v \in X_h(\Omega_k)$, we have

$$|\mathcal{M}_k v|^2_{H^1(\Omega_k)} \simeq |v|_{H_h^1(\Omega_k)}. \tag{6}$$

Proof For any $K \in \mathcal{T}_{h/2}(\Omega_k)$, there exists an element $E \in \mathcal{T}_h(\Omega_k)$ such that $K \subset E$. Assume that P_1 is the common vertex of K and E, P_2 and P_3 are other two vertices of K which lie on the edges of E, and P_4 is the fourth vertex of K which lies in E. Then

$$|\mathcal{M}_k v|^2_{H^1(K)} \simeq \sum_{i,j=1}^{4} |\mathcal{M}_k v(P_i) - \mathcal{M}_k v(P_j)|^2 \lesssim$$
$$\sum_{\tau \in \mathcal{T}_h(\Omega_k), P_1 \in \tau} \sum_{e_i, e_j \in \partial\tau} \left(\frac{1}{|e_i|}\int_{e_i} v\,\mathrm{d}s - \frac{1}{|e_j|}\int_{e_j} v\,\mathrm{d}s\right)^2 \lesssim$$
$$\sum_{\tau \in \mathcal{T}_h(\Omega_k), P_1 \in \tau} |v|^2_{H^1(\tau)}.$$

Summing over all $K \in \mathcal{T}_{h/2}(\Omega_k)$ yields

$$|\mathcal{M}_k v|_{H^1(\Omega_k)} \lesssim |v|_{H^1(\Omega_k)}. \tag{7}$$

For any $E \in \mathcal{T}_h(\Omega_k)$, let $e_i\,(i = 1, 2, 3, 4)$ be the edges of E, $Q_i\,(i = 1, 2, 3, 4)$ are the midpoints of $e_i\,(i = 1, 2, 3, 4)$, we derive

$$|v|^2_{H^1(E)} \simeq \sum_{i,j=1}^{4}\left(\frac{1}{|e_i|}\int_{e_i} v\,\mathrm{d}s - \frac{1}{|e_j|}\int_{e_j} v\,\mathrm{d}s\right)^2 =$$
$$\sum_{i,j=1}^{4} (\mathcal{M}_k v(Q_i) - \mathcal{M}_k v(Q_j))^2 \lesssim \sum_{K \subset E, K \in \mathcal{T}_{h/2}(\Omega_k)} |\mathcal{M}_k v|^2_{H^1(K)}.$$

Summing over all $E \in \mathcal{T}_h(\Omega_k)$ gives

$$|v|_{H_h^1(\Omega_k)} \lesssim |\mathcal{M}_k v|_{H^1(\Omega_k)}. \tag{8}$$

(6) follows from (7) and (8).

By interpolation estimate[5], the following result is easily to get

Lemma 2 $\| v - Q_{\delta} v \|_{L^2(\delta_{m(l)})} \lesssim h_l^{1/2} |v|_{H^{1/2}(\delta_{m(l)})}$, $\forall v \in H^{1/2}(\delta_{m(l)})$.

Moreover, for the operator I^{γ}, the following result is obvious.

Lemma 3 For any $v\in X_h(\Omega_k)$, then

$$\| Q_\delta I^\gamma Q_\gamma v|_{\gamma_{m(k)}} - I^\gamma Q_\gamma v|_{\gamma_{m(k)}} \|_{L^2(\gamma_{m(k)})} \preceq h_k^{1/2} |v|_{H^1(\Omega_k)},$$
$$\| I^\gamma Q_\gamma v|_{\gamma_{m(k)}} - Q_\gamma v|_{\gamma_{m(k)}} \|_{L^2(\gamma_{m(k)})} \preceq h_k^{1/2} |v|_{H^1(\Omega_k)}. \quad (9)$$

Proof From the definition of the operator $\mathscr{M}_k$ and I^γ, we can see

$$\mathscr{M}_k v = I^\gamma Q_\gamma v_k, \quad \text{on } \gamma_{m(k)}, \quad \forall v\in X_h(\Omega_k), \quad v_k = v|_{\gamma_{m(k)}}.$$

Using Lemma 1, Lemma 2, trace theorem and above equality, we deduce

$$\| Q_\delta I^\gamma Q_\gamma v|_{\gamma_{m(k)}} - I^\gamma Q_\gamma v|_{\gamma_{m(k)}} \|_{L^2(\gamma_{m(k)})} \preceq h_k^{1/2} |\mathscr{M}_k v|_{H^{1/2}(\gamma_{m(k)})} \preceq h_k^{1/2} |\mathscr{M}_k v|_{H^1(\Omega_k)} \preceq h_k^{1/2} |v|_{H^1(\Omega_k)}.$$

Hence, the first inequality holds, for the second inequality, we have

$$\| I^\gamma Q_\gamma v|_{\gamma_{m(k)}} - Q_\gamma v|_{\gamma_{m(k)}} \|_{L^2(\gamma_{m(k)})} = \| \mathscr{M}_k v - Q_\gamma v_k \|_{L^2(\gamma_{m(k)})} = \sum_{e\in \mathscr{T}^k_{h/2}(\Gamma_{kl})} \int_e (\mathscr{M}_k v - Q_e v_k)\,\mathrm{d}s, \quad (10)$$

where $Q_e v_k = Q_{2e} v_k$, $e\subset 2e \subset \mathscr{T}_h^k(\Gamma_{kl})$, and Q_{2e} is the L^2 orthogonal projection onto one dimensional space which consists of constant functions on element $2e$ on $\mathscr{T}_h^k(\gamma_{m(k)})$. Using the scaling argument in [5], for any constant c, we have

$$\int_e (\mathscr{M}_k v - c)^2 \mathrm{d}s \preceq h_k \int_{\hat{e}} (\mathscr{M}_k v - c)^2 \mathrm{d}\hat{s} \preceq h_k \| \mathscr{M}_k v - c \|^2_{1,\hat{E}} \preceq h_k |\mathscr{M}_k v|^2_{1,\hat{E}} \preceq h_k |\mathscr{M}_k v|^2_{1,E},$$

which together with (10) and Lemma 1, gives

$$\| I^\gamma Q_\gamma v|_{\gamma_{m(k)}} - Q_\gamma v|_{\gamma_{m(k)}} \|_{L^2(\gamma_{m(k)})} \preceq h_k^{1/2} |\mathscr{M}_k v|_{H_h^1(\Omega_k)} \preceq h_k^{1/2} |v|_{H_h^1(\Omega_k)}.$$

So we have complete the proof

3 Error Estimate

In this section, the error between the discrete solution of (5) and the solution of (2) is estimated

The following result is the well-known second Strang Lemma

Lemma 4 Let u and u_h be solution of (2) and (5) respectively, then

$$|u - u_h|_{H_h^1(\Omega)} \leqslant \inf_{v\in V_h} |u - v|_{H_h^1(\Omega)} + \sup_{\omega\in V_h} \left| \frac{\sum_{k=1}^N \sum_{E\in \mathscr{T}_h(\Omega_k)} \int_{\partial E} \frac{\partial u}{\partial n}\Omega \mathrm{d}s}{|\omega|_{H_h^1(\Omega)}} \right|. \quad (11)$$

The first term in (11) is known as the approximation error, while the second term is called the consistency error.

Lemma 5 Let u, u_h be the solution of (2) and (5) respectively, $u|_{\Omega_k}\in H^2(\Omega_k)$, then we have

$$\left| \sum_{k=1}^N \sum_{E\in \mathscr{T}_h(\Omega_k)} \int_{\partial E} \frac{\partial u}{\partial n}\omega \mathrm{d}s \right| \preceq \left(\sum_{k=1}^N h_k^2 |u|^2_{H^2(\Omega_k)} \right)^{\frac{1}{2}} |\omega|_{H_h^1(\Omega)}, \quad \forall \omega\in V_h.$$

Proof Note that

$$\sum_{k=1}^N \sum_{E\in \mathscr{T}_h(\Omega_k)} \int_{\partial E} \frac{\partial u}{\partial n}\omega \mathrm{d}s = \sum_{k=1}^N \sum_{E\in \mathscr{T}_h(\Omega_k)} \left(\sum_{e\in \partial E\setminus\Gamma} \int_{\partial E} \frac{\partial u}{\partial n}\omega \mathrm{d}s + \sum_{\Gamma_{kl}\in\Gamma} \int_{\Gamma_{kl}} \frac{\partial u}{\partial n}[\omega] \mathrm{d}s \right), \quad (12)$$

where e is an edge of E, $[\omega]$ denote the jump of ω across Γ_{kl}.

The first term in the right hand of (12) can be estimate by stand argument

$$\left| \sum_{k=1}^N \sum_{E\in \mathscr{T}_h(\Omega_k)} \sum_{e\in \partial E\setminus\Gamma} \int_{\partial E} \frac{\partial u}{\partial n}\omega \mathrm{d}s \right| \preceq \left(\sum_{k=1}^N h_k^2 |u|^2_{H^2(\Omega_k)} \right)^{\frac{1}{2}} |\omega|_{H_h^1(\Omega)}. \quad (13)$$

For any $\gamma_{m(k)} = \delta_{n(l)} = \Gamma_{kl}$, by the mortar condition

$$\begin{aligned}
\left| \int_{\Gamma_{kl}} \frac{\partial u}{\partial n}[\omega] \mathrm{d}s \right| = & \left| \int_{\Gamma_{kl}} \frac{\partial u}{\partial n} \{ (\omega_k - Q_\gamma \omega_k) + (Q_\gamma \omega_k - I^\gamma Q_\gamma \omega_k) + \right. \\
& \left. (I^\gamma Q_\gamma \omega_k - Q_\delta I^\gamma Q_\gamma \omega_k) + (Q_\delta \omega_l - \omega_l) \} \mathrm{d}s \right| \leqslant \\
& \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma_{kl})} \{ \| \omega_k - Q_\gamma \omega_k \|_{L^2(\Gamma_{kl})} + \| Q_\gamma \omega_k - I^\gamma Q_\gamma \omega_k \|_{L^2(\Gamma_{kl})} + \\
& \| I^\gamma Q_\gamma \omega_k - Q_\delta I^\gamma Q_\gamma \omega_k \|_{L^2(\Gamma_{kl})} + \| Q_\delta \omega_l - \omega_l \|_{L^2(\Gamma_{kl})} \} := \\
& \left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma_{kl})} (F_1 + F_2 + F_3 + F_4).
\end{aligned} \quad (14)$$

Trace theorem gives

$$\left\| \frac{\partial u}{\partial n} \right\|_{L^2(\Gamma_{kl})} \preceq h_l^{1/2} \, | u |_{H^2(\Omega_k \cup \Omega_l)}. \tag{15}$$

An application of the same argument as in the proof in (9) gives

$$F_1 \preceq h_k^{1/2} \, | \omega |_{H_h^1(\Omega_k)}, \quad F_2 \preceq h_k^{1/2} \, | \omega |_{H_h^1(\Omega_k)}, \quad F_4 \preceq h_l^{1/2} \, | \omega |_{H_h^1(\Omega_l)}.$$

For F_3, by Lemma 3, we have

$$F_3 \preceq h_k^{1/2} \, |\omega|_{H_h^1(\Omega_k)}.$$

Above four inequality and (14) yield

$$\left| \int_{\Gamma_{kl}} \frac{\partial u}{\partial n} [\omega] \, ds \right| \preceq h_l \, | u |_{H^2(\Omega_k \cup \Omega_l)} \left(| \omega |_{H_h^1(\Omega_k)} + | \omega |_{H_h^1(\Omega_l)} \right),$$

where the assumption $h_k \leqslant h_l$ is used. Summing over all $\Gamma_{kl} \in \Gamma$, using (12) and (13), the desired result follows.

Lemma 6 For any $u \in H_0^1(\Omega)$, $u|_{\Omega_k} \in H^2(\Omega_k)$, it holds that

$$\inf_{v \in V_h} | u - v |_{H_h^1(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^2 \, | u |_{H^2(\Omega_k)} \right)^{1/2}.$$

Proof Let $\Pi_h^k u \in X_h(\Omega_k)$ be the interpolation of u in $X_h(\Omega_k)$, we have the following estimate

$$\| u - \Pi_h^k u \|_{L^2(\Omega_k)} + h_k \, | u - \Pi_h^k u |_{H_h^1(\Omega_k)} \preceq h_k^2 \, | u |_{H^2(\Omega_k)}. \tag{16}$$

Define $\tilde{v}|_{\Omega_k} = \Pi_h^k u$, $k = 1, \ldots, N$. The function $\tilde{v} \in X_h(\Omega)$ may not satisfy the mortar condition across the interfaces. So we need to construct an operator $\Xi_{\delta_{m(l)}}: X_h(\Omega) \to X_h(\Omega)$ by

$$\int_e \Xi_{\delta_{m(l)}}(\tilde{v}) \, ds = \begin{cases} \int_e I^{\gamma} Q_{\gamma} \tilde{v}|_{\gamma_{m(k)}} - \tilde{v}|_{\delta_{m(l)}} \, ds, & e \in \mathcal{T}_h^l(\delta_{m(l)}) \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $\tilde{v} \in X_h(\Omega)$, let

$$v = \tilde{v} + \Xi_{\delta_{m(l)}}(\tilde{v}),$$

where the sum is taken over all nonmortars.

We can check that $v \in V_h$. From (16), we know

$$| u - v |_{H_h^1(\Omega)} \preceq | u - \tilde{v} |_{H_h^1(\Omega)} + | \Xi_{\delta_{m(l)}}(\tilde{v}) |_{H_h^1(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^2 \, | u |^2_{H^2(\Omega_k)} \right)^{1/2} + | \Xi_{\delta_{m(l)}}(\tilde{v}) |_{H_h^1(\Omega)}.$$

For each nonmortar edge $\delta_{m(l)}$, we can deduce

$$\begin{aligned} | \Xi_{\delta_{m(l)}} \tilde{v} |^2_{H_h^1(\Omega)} &= \sum_{E \in \mathcal{T}_h(\Omega)} \int_E | \nabla (\Xi_{\delta_{m(l)}} \tilde{v}) |^2 \, dx \preceq \sum_{E \in \mathcal{T}_h(\Omega)} \sum_{e \in \partial E} \left(\frac{1}{|e|} \int_e \Xi_{\delta_{m(l)}} \tilde{v} \, ds \right)^2 \preceq \\ & h_l^{-2} \sum_{e \in \mathcal{T}_h(\delta_{m(l)})} \left(\int_e Q_{\delta} (I^{\gamma} Q_{\gamma} \tilde{v}|_{\gamma_{m(k)}} - \tilde{v}|_{\delta_{m(l)}}) \, ds \right)^2 \preceq \\ & h_l^{-1} \left(\| I^{\gamma} Q_{\gamma} \Pi_h^k u - \Pi_h^l u \|^2_{L^2(\delta_{m(l)})} \right) \leqslant \\ & h_l^{-1} \left(\| I^{\gamma} Q_{\gamma} \Pi_h^k u - Q_{\gamma} \Pi_h^k u \|^2_{L^2(\Gamma_{kl})} + \| Q_{\gamma} \Pi_h^k u - \Pi_h^k u \|^2_{L^2(\Gamma_{kl})} + \right. \\ & \left. \| \Pi_h^k u - u \|^2_{L^2(\Gamma_{kl})} + \| \Pi_h^l u - u \|^2_{L^2(\Gamma_{kl})} \right), \end{aligned} \tag{17}$$

by a similar argument as in the proof in (9), trace theorem, Lemma 3 and (16), we obtain

$$\| I^{\gamma} Q_{\gamma} \Pi_h^k u - Q_{\gamma} \Pi_h^k u \|_{L^2(\Gamma_{kl})} \preceq h_k^{1/2} \| \Pi_h^k u \|_{H_h^1(\Omega_k)} \preceq h_k^{3/2} \, | u |_{H^2(\Omega_k)},$$

$$\| Q_{\gamma} \Pi_h^k u - \Pi_h^k u \|_{L^2(\Gamma_{kl})} \preceq h_k^{1/2} \| \Pi_h^k u \|_{H_h^1(\Omega_k)} \preceq h_k^{3/2} \, | u |_{H^2(\Omega_k)},$$

$$\| \Pi_h^k u - u \|_{L^2(\Gamma_{kl})} \preceq \left(h_k^{-1} \| u - \Pi_h^k u \|_{L^2(\Omega_k)}^2 + h_k \, | u - \Pi_h^k u |_{H_h^1(\Omega_k)}^2 \right)^{1/2} \preceq h_k^{3/2} \, | u |_{H^2(\Omega_k)},$$

$$\| \Pi_h^l u - u \|_{L^2(\Gamma_{kl})} \preceq \left(h_l^{-1} \| u - \Pi_h^l u \|_{L^2(\Omega_l)}^2 + h_l \, | u - \Pi_h^l u |_{H_h^1(\Omega_l)}^2 \right)^{1/2} \preceq h_l^{3/2} \, | u |_{H^2(\Omega_l)}.$$

Above four inequalities and (17) yield

$$| \Xi_{\delta_{m(l)}} \tilde{v} |_{H_h^1(\Omega)} \preceq h_k \, | u |_{H^2(\Omega_k)} + h_l \, | u |_{H^2(\Omega_l)}.$$

Summing over all nonmortars $\delta_{m(l)} \subset \partial \Omega_l$ and afterwards over the subdomains, we complete the proof.

From Lemma 4 – Lemma 6, we now state our conclusion concerning the error estimate.

Theorem 1 Let u, u_h be the solution of (2) and (5) respectively, $u|_{\Omega_k} \in H^2(\Omega_k)$, then

$$| u - u_h |_{H_h^1(\Omega)} \preceq \left(\sum_{k=1}^{N} h_k^2 | u |_{H^2(\Omega_k)}^2 \right)^{1/2}.$$

4 Numerical Experiments

In this section, we present the results of some numerical experiments which show that the mortar element method with rotated Q_1 element is optimal. For problem (1), let $\Omega = [0, 1] \times [0, 1]$, the domain Ω is divided into two adjacent subdomains, each subdomain is divided into a grid of smaller triangles, the meshes do not match on the interface. We assign $\Omega_1 = [0, 1] \times [0, 0.5]$ is mortar subdomain, $\Omega_2 = [0, 1] \times [0.5, 1]$ is non-mortar subdomain. Assume that the exact solution of (1) is $u = x(1-x)y(1-y)$.

By Gauss elimination method, we get the results as Table 1. In this table, $h_k (k = 1, 2)$ is the mesh size in $\mathcal{T}_h(\Omega_k)$, u_h denotes the solution of (5).

Table 1 The error between u and u_h

h_1^{-1}	h_2^{-1}	$\lvert u - u_h \rvert_{H_h^1(\Omega)}$
6	4	0.038 133 8
12	8	0.019 410 8
24	16	0.009 815 9
48	32	0.004 943 6
96	64	0.002 483 3
192	128	0.001 245 4

From Table 1, we can see the mortar element method coupling with rotated Q_1 element is optimal.

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