Optimal Investment and Proportional Reinsurance for Jump-Diffusion Risk Processes: Expected Value Principle

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Abstract: In this paper, we study, from the insurer's point of view, the optimal investment and proportional reinsurance for the jump-diffusion surplus processes. Assuming that the reinsurance premium is calculated according to the expected value principle, we obtain the closed form expressions of the strategy and the value function which are optimal in the sense of maximizing the expected utility from terminal wealth. We also conclude that the case with investment is always better than the one without investment. Some numerical examples are given, which illustrate the results of this paper.

Key words: stochastic control, Hamilton-Jacobi-Bellman equation, jump-diffusion process, expected utility, investment, proportional reinsurance, expected value principle

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跳扩散风险过程的最优投资和比例再保险:期望值保费原理

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[摘要] 站在保险人的立场上,讨论了期望值保费原理下,跳扩散风险过程的最优投资和比例再保险问题,得到了使终值期望效用达到最大的最优策略和值函数的近似表达式,并且得出结论:投资总比不投资好.最后,通过一些数值举例来进一步说明本文中所得的结论.

[关键词] 随机控制,Hamilton-Jacobi-Bellman方程,跳扩散过程,期望效用,投资,比例再保险,期望值原理

Recently, the optimization problem of an insurer has been studied extensively in the literature. See, for example, Browne[1], Hipp and Plum[2,3], Schmidli[4,5], Liu and Yang[6], Gerber and Shiu[7], Liang[8,9]. In these works, stochastic control theory and related tools have been widely used. They assume that the aggregate claims process is a compound Poisson process or a Brownian motion with drift, where the variables, such as, reinsurance, new businesses, investment and dividend, are adjusted dynamically. Under some assumptions, they are able to obtain closed-form solutions for the optimal strategy and the value function in the sense of maximizing (or minimizing) a certain objective function under different constraints. For example, Browne[1], Schmidli[4], Liang[8,9] consider one of (or both) the two controls to minimize the probability of ruin: (1) investing in a risky asset and (2) purchasing proportional reinsurance. They derive the explicit expression for the optimal values from the Brownian motion model. Yang and Zhang[10] discuss the optimal investment problem in the jump-diffusion model to maximize the expected utility. They also derive the closed-form expression for the optimal strategy

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and the optimal value function.

Although the literature on the optimal proportional reinsurance is increasing rapidly. See, for example, Waters[11], Centeno[12], Liang[8], Liang and Guo[13,14], Schmidli[4,5] and references therein. The most usual criterion of optimization is to maximize the adjustment coefficient or to minimize the probability of ruin. Very few of these contributions deal with the problem in relation to the criterion of maximizing the expected utility from terminal wealth. In this paper, assuming that the reinsurance premium is calculated according to the expected value principle, we consider the optimal investment and proportional reinsurance for the jump-diffusion risk process to maximize the expected utility from terminal wealth. The case with variance principle has been discussed in Liang[9]. The same problem with more complicated model is considered by Irgens and Paulsen[15], some explicit expressions for the value function are derived. However, there is no constraint on the controls which is different to this paper. Our contribution in this paper is to find the closed form expressions of the optimal strategy and the value function under the constraints on the two control variables. Further, we conclude that the case with investment is always better than the one without investment.

1 The Model and the Hamilton-Jacobi-Bellman (HJB) Equation

1.1 The model

Under the classical risk model, the surplus process $\{U_t\}_{t\geqslant 0}$ is given by

$$U_t = u + ct - S_t, \tag{1}$$

where $u\geqslant 0$ is the initial surplus, c is the premium rate and S_t represents the aggregate claims up to time t. We assume that $S_t = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process, i.e., $N(t)$ is a homogeneous Poisson process with intensity λ, Y_i, $i\geqslant 1$ is a sequence of positive i.i.d. random variables with common distribution $F(y)$, mean value $\mu = E(Y_i)$ and moment generating function $M_Y(r) = E\mathrm{e}^{rY_1}$. The claim number process $N(t)$ is also independent of the claim amount Y_i, $i\geqslant 1$. An introduction to classical ruin theory can be found, for instance, in Ref. [16].

In this paper, we consider the risk process which is perturbed by a Brownian motion, we also suppose that the insurer has the possibility to choose proportional reinsurance with level $q_t\in[0,1]$. The premium rate for the reinsurance is $(1+\eta)(1-q_t)\lambda\mu$, η is the safety loading of the reinsurer. Let $\theta=\frac{c}{\lambda\mu}-1$ be the safety loading of the insurer. Without loss of generality, we assume $\eta>\theta$. Then the surplus process of the insurer becomes

$$\mathrm{d}R_t^q = [q_t(1+\eta)-(\eta-\theta)]\lambda\mu\,\mathrm{d}t + \beta\,\mathrm{d}W_t - q_t\,\mathrm{d}S_t, \tag{2}$$

where $\beta\geqslant 0$ is a constant, W_t is a standard Brownian motion independent of the claim number process $N(t)$ and of Y_i, $i\geqslant 1$. The diffusion term βW_t represents the additional uncertainty associated with the insurance market or the economic environment. The uncertainty is not necessarily related to the claims, therefore, in this paper, we only consider the case where βW_t is not affected by reinsurance at all.

In Ref. [8], under the criterion of maximizing the expected utility from terminal wealth, the optimization problems of model (2), as well as its diffusion approximation model, have been discussed.

Here, we assume that there is one risky asset available for insurer in the financial market, whose price at time t is denoted by $P(t)$, and modeled as a geometric Brownian motion

$$\mathrm{d}P(t) = aP(t)\,\mathrm{d}t + \sigma P(t)\,\mathrm{d}B_t, \tag{3}$$

where a, σ are positive constants, and represent the expected instantaneous rate of return of the risky asset and the volatility of the risky asset, respectively. B_t, $t\geqslant 0$, is another standard Brownian motion independent of the claim number process $N(t)$ and of Y_i, $i\geqslant 1$.

Let $A_t\geqslant 0$ be the total amount of money invested in the higher risky asset and $X_t^{A,q}$ denote the wealth of the company at time t. If $X_t^{A,q}$ follows reinsurance strategy q as well as investment strategy A. This process then can be expressed as

$$dX_t^{A,q} = A_t(a\,dt + \sigma dB_t) + [q_t(1+\eta) - (\eta-\theta)]\lambda\mu dt + \beta dW_t - q_t dS_t. \tag{4}$$

Denote by ρ the correlation coefficient of B_t with W_t, i.e., $E[B_tW_t] = \rho t$.

Remark 1 In Ref. [13, 14], under the criterion of maximizing the adjustment coefficient, the optimal problems of models (2) and (4) have been discussed, some useful conclusions have been derived. However, the control variables A and q considered in [13, 14] are both constants, which are different from those in this paper.

Remark 2 In this paper, we assume that continuous trading is allowed, and all assets are infinitely divisible and we work on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which the process $X_t^{A,q}$ is well defined. The information at time t is given by the complete filtration F_t generated by $X_t^{A,q}$. The strategy (A_t, q_t) is $\mathcal{F}_t$-progressively measurable, and satisfies the integrability condition $\int_0^{\mathcal{T}} A_s^2 ds < \infty$, a.s. for all $\mathcal{T} < \infty$.

1.2 The HJB equation

In this paper, we assume that the insurer is interested in maximizing the utility from his terminal wealth, say at time T. The utility function is $u(x)$, where we assume that $u' > 0$ and $u'' < 0$. We want to maximize the utility attainable by the insurer from the state x at time t, the value function is then $V(t, x) = \sup_{q \in [0,1]} E[u(X_T^{A,q}) \mid X_t^{A,q} = x]$. While utility functions are subjective, there are of course specific utility functions that have many objective criteria associated with them. A variety of utility functions are studied in investment strategies, see Merton[17], Karatzas[18] for examples. Here, we assume that the insurer has an exponential utility function

$$u(x) = \lambda_1 - \frac{m}{\nu} e^{-\nu x},$$

where $m > 0$ and $\nu > 0$. This utility has constant absolute risk aversion (CARA) parameter ν. Such utility function plays an important role in insurance mathematics and actuarial practice, since they are the only functions under which the principle of "zero utility" gives a fair premium that is independent of the level of reserves of an insurance company[19].

We assume now that the value function $V(t, x)$ is twice continuously differentiable. Applying Ito's formula for jump-diffusion process[20], we obtain that the generator is

$$\mathcal{A}^{A,q}V(t, x) = V_t + [aA + ((1+\eta)q - (\eta - \theta))\lambda\mu]V_x + \frac{1}{2}[A^2\sigma^2 + \beta^2 + 2A\sigma\beta\rho]V_{xx} + \lambda E[V(t, x - qY) - V(t, x)], \tag{5}$$

where V_t, V_x and V_{xx} denote, respectively, the first order partial derivative with respect to t, the first order and the second order partial derivatives with respect to x. By a standard method[21], we can obtain the following HJB equation:

$$\sup_{(A,q)\in[0,\infty]\times[0,1]} \mathcal{A}^{A,q}V(t, x) = 0, \tag{6}$$

with the boundary condition $V(T, x) = u(x)$. The methods in Ref. [3] can be used to prove that the HJB equation has a classical solution in our case, and also the methods in Ref. [10] can be used to verify that this solution is just the optimal value function we want to get. Here we will not discuss these problems in detail.

2 The Optimal Investment and Proportional Reinsurance

In this section, we consider the problem of optimal investment and proportional reinsurance. Throughout this paper, we assume that $E(Ye^{rY}) = M'_Y(r)$ exists for $0 < r < \zeta$, and $\lim_{r\to\zeta} E(Ye^{rY}) = \infty$ for some $0 < \zeta \leqslant +\infty$. Furthermore, the Lundberg-Cramér conditions hold[16].

From (5) and (6), we derive the HJB equation, for $t \leqslant T$:

$$\sup_{A,q}\left\{ V_t + [aA + ((1+\eta)q - (\eta-\theta))\lambda\mu]V_x + \frac{1}{2}(\sigma^2A^2 + \beta^2 + 2\sigma A\beta\rho)V_{xx} + \lambda E[V(t, x - qY) - V(t, x)]\right\} = 0 \tag{7}$$

Now we assume that equation (7) has a classical solution V, which satisfies $V_x > 0$ and $V_{xx} < 0$. To solve the equation (7), inspired by Yang and Zhang[10], we try to fit a solution of form

$$V(t,x) = \lambda_1 - \frac{m}{\nu}\exp[-\nu x + h(T-t)], \tag{8}$$

where $h(\cdot)$ is a suitable function such that (8) is a solution of (7), and the boundary condition $V(T,x) = u(x)$ implies $h(0) = 0$.

From (8), we can get

$$\begin{cases} V_t = [V(t,x) - \lambda_1][-h'(T-t)], \\ V_x = [V(t,x) - \lambda_1][-\nu], \\ V_{xx} = [V(t,x) - \lambda_1][\nu^2], \\ E[V(t,x-qY) - V(t,x)] = [V(t,x) - \lambda_1][M_Y(\nu q) - 1]. \end{cases} \tag{9}$$

Replacing (9) back into equation (7) and canceling the like terms yield

$$\inf_{A,q}\{-h'(T-t) - [aA + ((1+\eta)q - (\eta-\theta))\lambda\mu]\nu + \frac{1}{2}(\sigma^2A^2 + \beta^2 + 2\sigma A\beta\rho)\nu^2 + \lambda[M_Y(\nu q) - 1]\} = 0. \tag{10}$$

Now, we need to find the optimal strategies A^* and q^*, which minimize the following function

$$f_2(A,q) = -h'(T-t) - [aA + ((1+\eta)q - (\eta-\theta))\lambda\mu]\nu + \frac{1}{2}(\sigma^2A^2 + \beta^2 + 2\sigma A\beta\rho)\nu^2 + \lambda[M_Y(\nu q) - 1]. \tag{11}$$

Differentiating with respect to A in (11) gives the optimizer

$$A_1 = \frac{a}{\sigma^2\nu} - \frac{\beta\rho}{\sigma}.$$

Since

$$\frac{\partial^2 f_2(A,q)}{\partial A^2} = \sigma^2\nu^2 > 0,$$

$f_2(A,q)$ is a convex function with regard to A, then the optimal investment strategy is

$$A^* = \begin{cases} 0, & \text{if } A_1 \leqslant 0, \\ \dfrac{a}{\sigma^2\nu} - \dfrac{\beta\rho}{\sigma}, & \text{if } A_1 > 0. \end{cases}$$

If $A^* = 0$, the optimal problem has been considered in Ref [8]. So, in the following context, we only consider the case with $A^* > 0$.

Now differentiating with respect to q in (11) yields a new equation

$$(1+\eta)\mu = E[Ye^{q\nu Y}] := M_Y(q\nu),$$

or equivalently

$$(1+\eta)\mu = M_Y(r), \tag{12}$$

where $r = q\nu$. Then we have

Lemma 1 Equation (12) has a unique positive root ρ.

Proof Let

$$f(r) = (1+\eta)\mu$$

and

$$g(r) = M_Y(r).$$

We have

$$g(0) = \mu,$$
$$g'(r) = E(Y^2e^{rY}) > 0,$$
$$g''(r) = E(Y^3e^{rY}) > 0.$$

This means that $g(r)$ is an increasing convex function.

Since $f(r) \equiv (1+\eta)\mu > g(0) = \mu$, $f(r)$ and $g(r)$ have a unique point of intersection at some $0 < r < 1$. That is, equation (12) has a unique positive root and the proof is complete.

Let q_1 denote the argument where (11) attains its minimum in $\mathbf{R}_+$, then $\rho = q_1 \nu$, i.e., $q_1 = \rho/\nu$. Since

$$\frac{\partial^2 f_2(A, q)}{\partial q^2} = \lambda \nu^2 E[Y^2 e^{\nu q Y}] > 0,$$

$f_2(A, q)$ is a convex function with regard to q, then the optimal policy q^* can be given as

$$q^* = q_1 \wedge 1.$$

Remark 3 The optimal reinsurance in this paper is just the same as the one in Theorem 3.1 of Ref [8]. That is to say, the action of investing does not affect the optimal reinsurance strategy.

Now inserting $A^* > 0$ and q^* into equation (10), we can get the following results: when $q^* = \rho/\nu < 1$, we have $h'(T-t) = Q_1(\nu)$. Integrating and setting $h(0) = 0$, we get

$$h(T-t) = Q_1(\nu)(T-t),$$

where

$$Q_1(\nu) = \lambda[M_Y(\rho) - 1] - (1+\eta)\lambda\mu\rho + (\eta - \theta)\lambda\mu\nu - \frac{a^2}{2\sigma^2} + \frac{a\beta\rho\nu}{\sigma} + \frac{1}{2}\beta^2(1-\rho^2)\nu^2.$$

When $q^* = 1$, $h'(T-t) = Q_2(\nu)$, using the same way, we can get

$$h(T-t) = Q_2(\nu)(T-t),$$

where

$$Q_2(\nu) = \lambda[M_Y(\nu) - 1] - (1+\theta)\lambda\mu\nu - \frac{a^2}{2\sigma^2} + \frac{a\beta\rho\nu}{\sigma} + \frac{1}{2}\beta^2(1-\rho^2)\nu^2.$$

So the following theorem is directly derived.

Theorem 1 Let ρ be the positive root of equation (12), $Q_1(\nu)$ and $Q_2(\nu)$ be given as above, then the optimal policies are the fixed constants

$$q^* = \left(\frac{\rho}{\nu}\right) \wedge 1, \tag{13}$$

$$A^* = \frac{a}{\sigma^2} \cdot \frac{1}{\nu} - \frac{\beta\rho}{\sigma}, \tag{14}$$

and the value function is

$$V(t, x) = \lambda_1 - \frac{m}{\nu}\exp[-\nu x + h(T-t)], \tag{15}$$

where

$$h(T-t) = \begin{cases} Q_1(\nu)(T-t), & \text{if } q^* < 1, \\ Q_2(\nu)(T-t), & \text{if } q^* = 1. \end{cases} \tag{16}$$

From theorem 1, we have

Theorem 2 Under the expected value principle, the case with investment is always better than the one without investment.

Proof Comparing (16) with (15) in Ref [8], we can see that, to prove Theorem 2, it is sufficient to consider functions h_1 and h_2 defined as

$$h_1(w) = -\frac{1}{2}w^2 + \beta\rho\nu w + \frac{1}{2}\beta^2(1-\rho^2)\nu^2,$$

and

$$h_2(w) = \frac{1}{2}\beta^2\nu^2.$$

Here $w = \frac{a}{\sigma}$ is the return of the unit risk, a very important factor in risky investment. We know that if $h_1(w) \leqslant$

$h_2(w)$ for all $w \geqslant 0$, then $V_2(t, x) \geqslant V_1(t, x)$ for all $(t, x) \in \mathbf{R}^2$.

Note that

$$h_1(w) - h_2(w) = -\frac{1}{2}w^2 + \beta\rho\nu w - \frac{1}{2}\beta^2\rho^2\nu^2 = -\frac{1}{2}(w^2 - 2\beta\rho\nu w + \beta^2\rho^2\nu^2) = -\frac{1}{2}(w - \beta\rho\nu)^2 \leqslant 0$$

So we have $h_1(w) \leqslant h_2(w)$ for all $w \geqslant 0$, and thus $V_2(t, x) \geqslant V_1(t, x)$. This completes the proof.

Remark 4 Comparing the results in Theorem 1 with those in Ref. [8], we can see that whether the insurance company invests some capital to the stock market, the optimal reinsurance strategy is the same fixed constant. That is to say, the optimal reinsurance strategy is only dependent on the claim sizes, the risk attitude of the investor and the premium principle under which the insurer pays the premium to the reinsurer. Meanwhile, whether the insurance company carries out the reinsurance strategy has no effect on the investment strategy. Moreover, the optimal strategy A^* only depends on the five parameters a, σ, ν, β, ρ and has no relevance to the parameters λ, μ, θ, η, which means that the optimal investment strategy only depends on the external economic environment (β, ρ), the stock market (a, σ^2), and the risk attitude of the investor ν. It does not depend on the claim sizes, safety loading parameters or something else. That's really an amazing result. In fact, in the optimization problem of maximizing the adjustment coefficient, these two strategies do affect each other, see, for example, Liang and Guo[14].

Remark 5 The conclusion in Theorem 2 has also been derived in Ref. [9, 14].

3 Numerical Examples

In this subsection, we assume that the claim sizes Y are exponentially distributed with parameter $1/\mu = 1$. Some numerical examples are given to illustrate the results of this paper.

Lemma 2 Assume that the claim sizes are exponentially distributed with parameter $1/\mu = 1$. Then the positive solution of equation (12) is given as

$$\rho = 1 - \sqrt{\frac{1}{1+\eta}}. \tag{17}$$

Example 1 Let $\eta = 0.3, 0.6, 0.9$, we investigate the effect of ν on the optimal reinsurance strategy. The results are shown in Fig. 1.

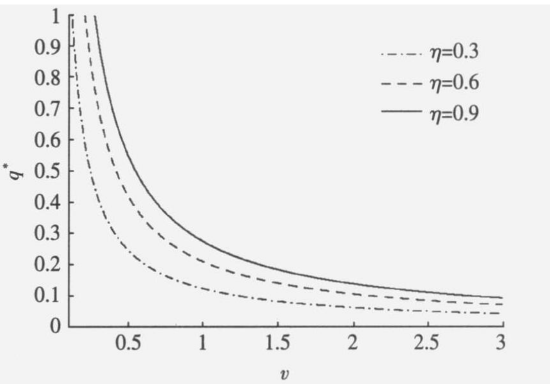

Fig.1 The effect of ν on q*

From Fig. 1, we find that the optimal reinsurance strategy decreases when the CARA parameter ν increases, which means that the less risk averse an insurer is, the larger his share in the risk will be. We can also see that, a larger value of η will yield a higher retention level of optimal reinsurance. This simply states that as the price of reinsurance increases, the insurer should retain a greater share of each claim.

Example 2 Let $\nu = 1$, $\beta = 0.01$, $\rho = 0.6$ and $a = 0.5, 1, 2$, we give the influences of σ on the optimal investment strategy A^*. The results are shown in Fig. 2.

From Fig. 2, we can conclude that a larger volatility of the stock market will yield a less investment in the risky asset. Whereas, a larger expected instantaneous rate of return of the risky asset yields a greater investment in the risky asset. All this results are just consistent with the practical market.

Example 3 Let $\theta = 0.3$, $\eta = 0.4$, $\beta = 0.1$, $\rho = 0.6$, $\lambda = 3$, $\lambda_1 = 3$, $m = 0.5$, $T = 10$ and $w = 1, 2$, we give the influences of ν on the value function $V(9, 1)$. The results are shown in Fig. 3.

From Fig. 3, we can see that the value function increases with the increasing of the parameter ν, and a larger return of the unit risk w will yield a larger expected utility. Further, we find that the expected utility with investment is larger than the one without investment, which is the natural consequence of Theorem 2.

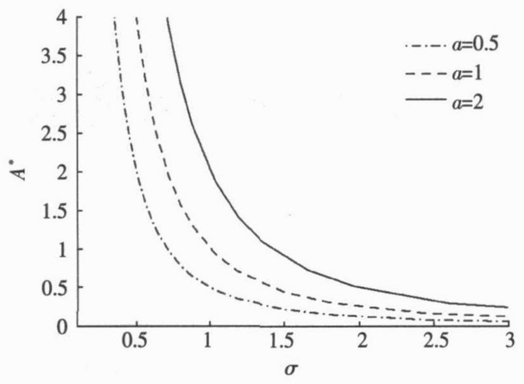

Fig.2 The effect of σ on A^*

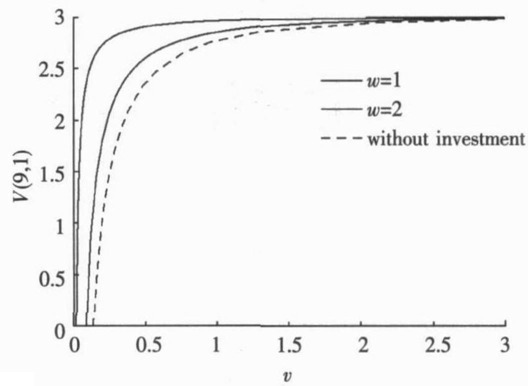

Fig.3 The effect of v on $V(9,1)$

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