Continuous Pre-ordered Sets

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Abstract The concept of continuous pre-ordered sets and its basic properties are introduced. It is showed that the category CPOSET of continuous posets and Scott continuous functions is a reflective subcategory of the category CPRSET of continuous pre-ordered sets and Scott continuous functions.

Key words: continuous pre-ordered sets, Scott continuous functions, small category

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连续预序集

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[摘要] 介绍了连续预序集的概念及其基本性质, 得到以连续偏序集为对象, Scott连续函数为态射的范畴 CPOSET是以连续预序集为对象, Scott连续函数为态射的范畴 CPRSET的反射子范畴.

[关键词] 连续预序集, Scott连续函数, 小范畴

Let L be a pre-ordered set, $a, b\in L, X\subseteq L$. We say that a is a lower bound of X (b is a upper bound), provided that $a\leqslant x$ for all $x\in X$ ($x\leqslant b$ for all $x\in X$). The set of all lower bounds of X is written as X^l (the set of all upper bounds of X is written as X^u). If the set of upper bounds of X has minimal elements, we call these elements the least upper bound and write it as $\wedge X$. Clearly $\vee X = D^u\cap D^{ul}$. Similarly the greatest lower bound is written as $\wedge X$. We know that $\vee X$ and $\wedge X$ may have more than one element. We write $\downarrow X = \{y\in L: y\leqslant x$ for some $x\in X\}$ and $\uparrow X = \{y\in L: x\leqslant y$ for some $x\in X\}$. The following results are straightforward[1].

Lemma 1 For a subset X of a pre-ordered set L, the following conditions are equivalent:

(i) X is directed;

(ii) $\downarrow X$ is directed;

(iii) $\downarrow X$ is an ideal.

Lemma 2 The following conditions are equivalent for L and X as in Lemma 1:

(i) $\vee X\neq \approxeq$;

(ii) $\vee\downarrow X\neq \approxeq$.

If these conditions are satisfied, then $\vee X = \vee\downarrow X$. Moreover, if every finite subset of X has least upper bound and if F denotes the set of all these finite least upper bound, then F is directed, and (i), (ii) are equivalent to

(iii) $\vee F\neq \approxeq$.

Under these circumstances, $\vee X = \vee F$.

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1 Continuous Pre-ordered Sets

Definition 1 Let L be a pre-ordered set. We say that x is way-below y, in symbols $x \ll y$, if and only if for all directed subsets $D \subseteq L$ with $\bigvee D \neq \approxeq$, the relation $y \in D^{ul}$ always implies the existence of a $d \in D$ with $x \leqslant d$. An element satisfying $x \ll x$ is called compact.

By Lemma 1 and Lemma 2, we know that directed subsets D in Definition 1 can be substituted by ideals I and the way-below relation can be defined equivalently by the following property:

$x \ll y$ if and only if for every subset $X \subseteq L$, the relation $y \in X^{ul}$ always implies the existence of a finite subset $A \subseteq X$ such that $x \in A^{ul}$.

Proposition 1 In a pre-ordered set L the following statements hold for all $u, x, y, z \in L$:

(i) $x \ll y$ implies $x \leqslant y$;

(ii) $u \leqslant x \ll y \leqslant z$ implies $u \ll z$;

(iii) $x \ll z$ and $y \ll z$ imply $u \ll z$ for all $u \in x \vee y$, if $x \vee y$ exists in L.

Definition 2 A pre-ordered set L is called continuous if it satisfies the following conditions:

(C_1) For all $x \in L$, the set $\Downarrow x = \{y \in L: y \ll x\}$ is directed in L;

(C_2) $x \in (\Downarrow x)^{ul}$, for all $x \in L$.

Example 1 Let X be a set with $|X| \geqslant \aleph_0$. 2^X is the power set of X. Define a pre-order on 2^X as following: $U, V \subseteq 2^X$, $U \leqslant V \Leftrightarrow |U| \leqslant |V|$.

The cardinality of U is denoted by $|U|$. Then $U < V$ obviously implies $U \ll V$. Conversely, if $U \ll V$, then either $U < V$, if $|V| \geqslant \aleph_0$ or $U \leqslant V$, if $|V| < \aleph_0$. Thus 2^X is a continuous pre-ordered set. This example also implies that $U \ll V \Rightarrow U \leqslant V$, but the converse is not true.

Proposition 2 (i) In a pre-ordered set L, the following conditions are equivalent:

(1) $x \ll y$;

(2) $x \in I$ for every ideal I of L such that $y \in I^{ul}$;

(3) $x \in \bigcap J(y)$, $J(y) = \{I \in IdL: y \in I^{ul}\}$.

(ii) Suppose that there exists a directed set $D \subseteq \Downarrow x$ with $x \in D^{ul}$. Then $\Downarrow x$ is directed and $x \in (\Downarrow x)^{ul}$. Furthermore, $y \ll x$ iff $y \ll x$ in the pre-ordered set $\downarrow x$ with the induced pre-order.

Proof (i) (1) $\Rightarrow$ (2) Clearly.

(2) $\Rightarrow$ (1): Assume (2) and let D be a directed subset of L with $\bigvee D \neq \approxeq$ and $y \in D^{ul}$. Then $I = \downarrow D$ is an ideal and $y \in D^{ul} = I^{ul}$. Then $x \in I$ by (2), i. e. there is a $d \in D$ such that $x \leqslant d$. Hence $x \ll y$.

Condition (3) is just a reformulation of (2).

(ii) Let $y \ll x$, $z \ll x$. Then $y \leqslant d_y$ and $z \leqslant d_z$ for some $d_y, d_z \in D$. Pick $d \in D$ such that $d_y, d_z \leqslant d$. Then $y \leqslant d$, $z \leqslant d$ and $d \ll x$. Thus $\Downarrow x$ is directed. If $y \ll x$ in $\downarrow x$ and $x \in (\Downarrow x)^{ul}$, then $y \leqslant d$ for some $d \in \Downarrow x$, so $y \ll x$ in L by Proposition 1 (ii).

Proposition 2 gives a sufficient condition for a pre-ordered set L to be continuous. And if L is a continuous pre-ordered set, $y \ll x$ in $\downarrow x \Leftrightarrow y \ll x$ in L.

We say that the way-below relation on a pre-ordered set L satisfies the strong interpolation property, provided that the following condition is satisfied for all $x, z \in L$:

(SI) $x \ll z$ and $z \notin (\Downarrow x)^{ul} \Rightarrow \exists\, y$, $x \ll y \ll z$, $y \notin (\Downarrow x)^{ul}$.

We say that the way-below relation on a pre-ordered set L satisfies the interpolation property if and only if the following weaker condition holds for all $x, z \in L$:

(INT) $x \ll z$ implies $(\exists\, y) x \ll y \ll z$.

Clearly, (SI) implies (INT); if L is a continuous pre-ordered set, then both conditions are equivalent.

Theorem 1 Let L be a continuous pre-ordered set, the following conditions hold for all $x, z \in L$:

(i) If $x \ll z$, $z \notin (\Downarrow x)^{ul}$ and $z \in D^{ul}$ for a directed subset D of L, then $x \ll d$, $d \notin (\Downarrow x)^{ul}$ for some element d

$\in D$.

(ii) If $x \ll z$, $z \notin (\Downarrow x)^{ul}$, then there exists a y such that $x \ll y \ll z$ and $y \notin (\Downarrow x)^{ul}$.

Proof (i) Let D be a directed set such that $z \in D^{ul}$, and let $I = \bigcup \{\Downarrow d: d \in D\}$. As L is a continuous pre-ordered set, I is an ideal as a directed union of the ideals $\Downarrow d$ and $I^{ul} = D^{ul}$. From $x \ll z$, we now conclude that $x \in I$, that is, there is an element $d \in D$, such that $x \ll d$. As $z \notin (\Downarrow x)^{ul}$ and $z \in D^{ul}$, there is an element $c \in D$ with $c \notin (\Downarrow x)^{ul}$. Replacing d by a common upper bound of c and d in D, we have found the desired element.

(ii) Choose $D = \Downarrow z$. As L is a continuous pre-ordered set, D is directed and $z \in D^{ul}$. If $x \ll z$ and $z \notin (\Downarrow x)^{ul}$, by (i), we find an element $y \in D$, that is, $x \ll y \ll z$ and $y \notin (\Downarrow x)^{ul}$.

Proposition 3 Let $\{L_i: i \in I\}$ be a family of continuous pre-ordered sets with minimum element 0_i for each L_i, then the direct product $\prod_{i \in I} L_i$ is also a continuous pre-ordered set. For elements $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ in $\prod_{i \in I} L_i$, the way-below relation is given by $x \ll y$ if and only if $x_i \ll y_i$ for all $i \in I$ and $x_i = 0_i$ for all but finitely many $i \in I$.

A function $p: L \to L$ is idempotent if and only if $pp = p$. A projection operator (shortly projection) is an idempotent, monotone self-map $p: L \to L$. A closure operator is a projection c on L with $1_L \leqslant c$. A kernel operator is a projection k on L with $k \leqslant 1_L$.

Lemma 3 For a projection p on a pre-ordered set L, consider its image $p(L)$ in L with the induced pre-order. Then the following properties hold:

(i) If X is a subset of $p(L)$ which $\vee X$ exists in L, then $\vee X$ exists in $p(L)$ and $p(\vee_L X) \subseteq \vee_{p(L)} X$.

(ii) In addition, if p preserves directed least upper bound, i.e. $\vee f(D)$ exists in L and $f(\vee D) = \vee f(D)$ for every directed set $D \subseteq L$ with $\vee D \neq \approxeq$, then for every directed subset $D \subseteq p(L)$ with $\vee_L D \neq \approxeq$, $\vee_{p(L)} D \neq \approxeq$ and $\vee_L D = \vee_{p(L)} D$.

Proof (i) Let $X \subseteq p(L)$ and $\vee_L X \neq \approxeq$. From $x \in (\vee_L X)^l$, we deduce that $x = p(L) \in [p(\vee_L X)]^l$ for every $x \in X$ by the monotonicity and the idempotency of p. So $p(\vee_L X) \subseteq X^u_{p(L)}$. For every $a \in X^u_{p(L)}$, we have $a \in X^u_L$. Then $a = p(a) \in [p(X_L)]^u$. Hence $p(\vee_L X) \subseteq X^{ul}_{p(L)}$. So $p(\vee_L X) \subseteq \vee_{p(L)} X$.

(ii) If $D \subseteq p(L)$ is directed and $\vee_L D \neq \approxeq$, then by (i), $p(\vee_L D) \subseteq \vee_{p(L)} D$. If p preserves directed least upper bound, then by idempotence $p(\vee_L D) = \vee_L p(D) = \vee_L D$ and $\vee_{p(L)} D \subseteq p(\vee_L D)$. Hence $\vee_L D = \vee_{p(L)} D$.

Example 2 In Lemma 3(i) the converse inclusion is not true. For example, L is a pre-ordered set (see Fig 1) and $p: L \to L$ is a projection

$$p(x) = \begin{cases} x, & \text{if } x \in X \cup \{a, b\}; \\ b, & \text{if } x = y. \end{cases}$$

Then $\vee_{p(L)} X = \{a, b\}$ ¾ $p(\vee_L X) = \{b\}$.

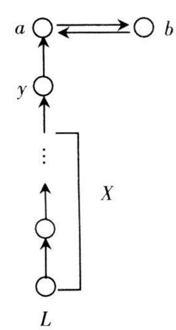

Fig.1 Set L

Theorem 2 Let L be a continuous pre-ordered set and $p: L \to L$ a projection and preserves directed least upper bound. Then the image $p(L)$ with the pre-order induced from L is a continuous pre-ordered set, too. For $x, y \in p(L)$, we have:

$x \ll_{p(L)} y$ if and only if there is an element $u \in L$ such that $x \leqslant p(u)$ and $u \ll_L y$.

Proof Let $y \in p(L)$ be given. As L is continuous, the set $\Downarrow_L y$ is directed and $p(y) \in \vee_L p(\Downarrow_L y)$. As $y \in p(L)$, we have $y = p(y)$. By Lemma 3, we have $y \in \vee_{p(L)} p(\Downarrow_L y)$. Thus it suffices to prove that $p(u) \ll_{p(L)} y$ whenever $u \ll_L y$ for the continuity of $p(L)$. For this, let u be an element of L such that $u \ll_L y$. Consider any directed subset $D \subseteq p(L)$ such that $y \in D^{ul}_{p(L)} = D^{ul}_L$. As $u \ll_L y$, we find a $d \in D$ such that $u \leqslant d$. Then $p(u) \leqslant p(d) = d$ by the monotonicity and idempotency of p. This shows that $p(u) \ll_{p(L)} y$. For the second part of the claim, suppose $x, y\in p(L)$ and $x\ll_{p(L)} y$. As $y\in D_{p(L)}^{ul}=D_L^{ul}$ by the above, there is a $u\in L$ with $u\ll_L y$ such that $x\leqslant p(u)$. The converse has already been shown in the first part of the proof.

As kernel and closure operators are particular kinds of projections, the previous results hold for images under kernel and closure operators provided that they preserve directed least upper bound. The characterization of the way- below relation on the image can be simplified as follows:

Remark 1 For a kernel operator k and a closure operator c on a continuous pre-ordered set L both preserving directed least upper bound, we have:

(i) For all $x, y\in k(L)$, one has $x\ll_{k(L)} y$ if and only if $x\ll_L y$.

(ii) For all $x, y\in L$, one has $x\ll_L y\Rightarrow c(x)\ll_{c(L)} c(y)$.

2 Scott Topology and Scott Continuous Functions on Pre-ordered Sets

Definition 3 A subset W of a pre-ordered set L is called Scott closed if for every directed set $D\subseteq W$ with $\vee D\neq\varnothing$ in L, we have $D^{ul}\subseteq W$. The complement of a Scott closed set is called a Scott open set. The collection of all Scott open subsets on L will be called the Scott topology of L and will be denoted by $\sigma(L)$.

Remark 2 (i) A Scott closed set is a lower set.

(ii) $cl_\sigma\{x\}=\downarrow x$. $\sigma(L)$ is not a T_0 topology.

(iii) A set U is Scott opened if and only if it is an upper set and $\varnothing\neq\vee D\subseteq U$ implies $D\cap U\neq\varnothing$ for all directed sets $D\subseteq L$.

(iv) If L is a continuous pre-ordered set, then all set $\Uparrow x\,(x\in L)$ are Scott open.

(v) A subset X of a pre-ordered set L has the property (S) provided that the following condition is satisfied:

If $\varnothing\neq\vee D\subseteq X$ for any directed set D, then there is a $y\in D$ such that $x\in X$ for all $x\in D$ with $y\leqslant x$.

A set is Scott open if and only if it is an upper set satisfying (S).

Proposition 4 Let L be a continuous pre-ordered set.

(i) An upper set U is Scott open if and only if for every $x\in U$ there is a $u\in U$ such that $u\ll x$.

(ii) The set of the form $\Uparrow u$, $u\in L$ form a basis for the Scott topology. In particular, each point $x\in L$ has a $\sigma(L)$ neighborhood basis consisting of the set $\Uparrow u$ with $u\ll x$.

Proof (i) The necessity is immediately from Definition 2 and Remark 2 (iii). Conversely if for every $x\in U$ there is a $u\in U$ such that $u\ll x$, then U is union of the sets $\Uparrow u$, $u\in U$, which is Scott open by Remark 2 (iv), hence U is Scott open.

(ii) is immediate consequence of (i).

Theorem 3[2] Let $f: M\to Z$ be a function between pre-ordered sets. The following conditions on f are equivalent.

(1) The inverse image of each Scott closed subset of Z is Scott closed;

(2) The inverse image of each principle ideal of Z is Scott closed;

(3) For each directed subset D of M such that $\vee D\neq\varnothing$, we have $f(D^{ul})\subseteq(fD)^{ul}$;

(4) For each directed subset D of M such that $\vee D\neq\varnothing$, we have $f(\vee D)\subseteq\vee(fD)$.

Definition 4 A function $f: S\to T$ between pre-ordered sets is Scott continuous if and only if it satisfies the equivalent conditions Theorem 3 (1), (2), (3), (4). The category whose objects are continuous pre-ordered sets and whose morphisms are Scott continuous functions will be denoted by CPRSET.

Example 3 A Scott continuous function $f: M\to Z$ between pre-ordered sets does not necessarily preserve directed least upper bound. For example, let M be a pre-ordered set (described in Fig 2) and $f: M\to M$ is a Scott continuous function: $f(i)=i$, $i=3, 4, 5$; $f(1)=f(2)=2$. Clearly M is a directed set and $\vee M=\{1, 2\}$. But $f(\vee M)=\{2\}\neq\vee f(M)=\{1, 2\}$.

Theorem 4 The category POSET (objects are posets and morphisms are Scott continuous functions) is a reflective subcategory (see [3] Proposition 3.3.6) of the category PRSET (objects are pre-ordered sets and morphisms are Scott continuous functions). In addition, for a pre-ordered set Q, $x \ll y \Leftrightarrow [x] \ll [y]$ for every $x, y \in Q$. Thus the category CPOSET (objects are continuous posets and morphisms are Scott continuous functions) is a reflective subcategory of the category CPRSET.

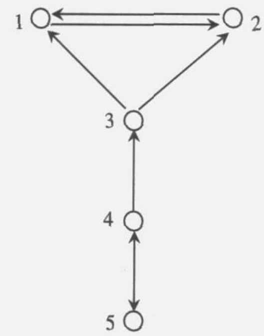

Fig.2 Set *M*

Proof Define $[x] = \{y \in Q : y \leqslant x \text{ and } x \leqslant y\}$ for every pre-ordered set Q and $x \in Q$. Then $F(Q) = \{[x] : x \in Q\}$ is a poset. Suppose $D \subseteq Q$ is a directed set and $\bigvee D \neq \varnothing$ in Q. We have known that $[x] = [y]$ for every $x, y \in \bigvee D$, so $r(\bigvee D) = [x]$ for some $x \in \bigvee D$. Clearly, $[x]$ is a upper bound of $r[D]$. For every $[t] \in F(Q)$ with $[d] \leqslant [t]$ for every $d \in D$, if $[x] \nleqslant [t]$, then $x \nleqslant t$ and $x \notin D^{ul}$ because of $t \in D^{u}$. There is a contradiction with $x \in \bigvee D$. Thus $[x] \leqslant [t]$. So $[x] = \bigvee_{d \in D} [d] = \bigvee r[D]$ i.e. $r : Q \to F(Q)$ $(x \mapsto [x])$ is Scott continuous.

For every poset P and every morphism f: $Q \to P$, we define $\bar{f}$: $F(Q) \to P([x] \mapsto f(x))$. As f is Scott continuous (order preserving), $f(x) = f(y)$ for every $x, y \in [x]$. Suppose $D' \subseteq F(Q)$ is a directed set and $\bigvee D'$ exists in $F(Q)$. Then $D = \cup \{[d'] : d' \in D'\}$ is a directed set in Q and $\bigvee D = [\bigvee D']$ in Q. As $\bar{f}(\bigvee D')$ and $\bigvee \bar{f}(D')$ contain one element respectively and $\bar{f}(\bigvee D') = f(\bigvee D) \subseteq \bigvee f(D) = \bigvee_{d' \in D'} \bar{f}(d') = \bigvee \bar{f}(D')$ by the definition of $\bar{f}$ and the Scott continuity of f, $\bar{f}(\bigvee D') = \bigvee \bar{f}(D')$ i.e. $\bar{f}$ is Scott continuous. Thus $\bar{f}$ is unique and $f = \bar{f} \circ r$.

Suppose $x \ll y$ in Q and $[y] \leqslant \bigvee D$ for every directed set $D \subseteq F(Q)$. $D' = \cup \{[d] : d \in D\}$ is a directed set in Q and $y \in \bigvee D'$. As $x \ll y$, there exists $d \in D'$ with $x \leqslant d$. Thus $[x] \leqslant [d]$ and $[x] \ll [y]$. Conversely, for every directed set $D' \subseteq Q$ with $y \in D'^{ul}$, $D = \cup \{[d'] : d' \in D'\}$ is a directed set in $F(Q)$ with $\bigvee D = [\bigvee D']$ and $[y] \leqslant [\bigvee D'] = \bigvee D$. As $[x] \ll [y]$, there exists $[d'] \in D$ with $[x] \leqslant [d']$. Thus $x \leqslant d'$ and $x \ll y$.

3 Categorical Viewpoint

In [3] and [4], we know every pre-ordered set $(X, \leqslant)$ gives rise to a small category $(X, \leqslant)$ as following:

Objects are elements of X, $\mathrm{Hom}(x, y) = \begin{cases} \{(x, y)\}, & \text{if } x \leqslant y; \\ \varnothing, & \text{otherwise.} \end{cases}$

A categorical product (coproduct) $\prod A_i (\coprod A_i)$ of a family of objects $\{a_i\}_{i \in I}$ in $(X, \leqslant)$ is $\bigwedge \{a_i\}_{i \in I} (\bigvee \{a_i\}_{i \in I})$.

Theorem 5[3] If a small category has products (or coproducts), then for every pair of objects A, B, morphism set $\mathrm{Hom}(A, B)$ has at most one element. Thus every small category which has products (or coproducts) is isomorphic to a pre-ordered set in which greatest lower bound (least upper bound) of every family of elements exists.

By Theorem 5, we know that all the propositions for pre-ordered sets in which greatest lower bound (least upper bound) of every family of elements exists are still valid for small categories which have coproducts (products).

Proposition 5 Let L be a small category. It has products if and only if it has coproducts.

Proof Suppose L has coproducts. A is an arbitrary subset of L. Let $B = \{b \in L : \exists f_i : b \to a_i, \text{ for every } a_i \in A\}$. As L has coproducts, $\coprod B$ exists. We say $\coprod B$ is a product of A. For every $a_i \in A$, there exists a unique morphism $r_i : \coprod B \to a_i$ with $f_j = r_i \circ q_j$ for every j. For every family of morphisms $\{p_i : Q \to a_i : i \in I\}$, then $Q \in B$

by the definition of B. Thus there is a unique morphism $r = q: Q \rightarrow \coprod B$ with $p_i = q \circ r_i$ for every $i \in I$. The converse can be proved similarly.

Now we consider Scott continuous functions on small categories with coproducts.

Definition 5 Let L be a small category which has coproducts and A a full subcategory of L. A is called Scott closed subcategory if and only if for every subset $D \subseteq A$, we have $\coprod D \in A$.

Definition 6 Let M and L be small categories which have coproducts. A functor $F: M \rightarrow L$ is called Scott continuous if and only if for every Scott closed subcategory $A \subseteq L$, $F^{-1}(A)$ is a Scott closed subcategory of M.

Theorem 6 If M and L are small categories which have coproducts, then a functor $F: M \rightarrow L$ is called Scott continuous if and only if F preserves coproducts.

Proof "$\Rightarrow$" Supposes an arbitrary subset A of L and $\coprod A$ exists. We have to prove that $F(\coprod A)$ is a product of $F(A)$. For every family of morphisms $\{f_i: F(a_i) \rightarrow Q: a_i \in A, i \in I\}$, there exists a unique morphism $r: \coprod F(A) \rightarrow Q$ with $f_i = r \circ q_i$ for every $i \in I$, $q_i: F(A) \rightarrow \coprod F(A)$. In particular, for a family of morphisms $\{p_i: a_i \rightarrow \coprod A: i \in I\}$, we have a unique morphism $r_1: \coprod F(A) \rightarrow F(\coprod A)$ such that $F(p_i) = r_1 \circ q_i$ for every $i \in I$. Thus there exists a unique morphism $r': F(\coprod A) \rightarrow Q$ such that $f_i = r' \circ F(p_i)$ for every $i \in I$.

"$\Leftarrow$" For every Scott closed subcategory $A \subseteq L$, we suppose that D is an arbitrary subset of $F^{-1}(A)$. Then $F(D) \subseteq A$ and $\coprod F(D) \subseteq A$. As F preserves coproducts, we have $F(\coprod D) \subseteq A$. Thus $\coprod D \subseteq F^{-1}(A)$. $F^{-1}(A)$ is a Scott closed subcategory and F is Scott continuous.

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