Ishikawa Iterative Algorithm for a System of g-η-Accretive Mapping Inclusions in Banach Spaces

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Abstract In this paper, we introduce and study a new system of g-η-accretive mapping inclusions in Banach spaces. Using the resolvent operator technique for g-η-accretive mappings, we prove the existence and uniqueness of the solutions for this system of g-η-accretive mapping inclusions. We also construct a new Ishikawa iterative algorithm for solving this system of g-η-accretive mapping inclusions and discuss the convergence of iterative sequence generated by the algorithm.

Key words Ishikawa iterative algorithm, system of g-η-accretive mapping inclusions, ralaxed cocoercive mapping, resolvent operator technique, existence, convergence

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Banach空间中一类 g-η-增生映象包含组的 Ishikawa迭代算法

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[摘要] 在 Banach空间中引入和研究了一类含 g-η-增生映象的变分包含, 利用与 g-η-增生映象相联系的预解算子技巧, 证明了这类 g-η-增生映象包含解的存在性和惟一性. 对这类 g-η-增生映象包含的逼近解我们也构造了一个新的 Ishikawa迭代算法, 并讨论了由此算法生成的迭代序列的收敛性.

[关键词] Ishikawa迭代算法, g-η-增生映象包含组, 松驰余制映象, 预解算子技巧, 存在性, 收敛性

1 Variational Inclusions and Preliminaries

Variational inclusions problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusions problems have been studied. For details, we can refer to [1-6] and the references therein.

In the present work, our study is motivated by Refs [1-4]. In this paper, we shall introduce and study a new system of g-η-accretive mapping inclusions in Banach spaces. Using the resolvent operator technique for g-η-accretive mappings, we prove the existence and uniqueness of the solutions for this system of g-η-accretive mapping inclusions. We also construct a new Ishikawa iterative algorithm for solving this system of g-η-accretive

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mapping inclusions and discuss the convergence of iterative sequence generated by the algorithm. The results presented in this paper generalize and improve the recent corresponding results variational inequalities (inclusions) problems in Refs[1–4].

Throughout this paper, let X be a real Banach space with dual space X^*, $\langle \cdot , \cdot \rangle$ be the dual pair between X and X^*, 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q: X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^*: \langle x, f^* \rangle = \|x\|^q, \ \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant.

Lemma 1[7] Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $C_q > 0$, such that for all $x, y \in X$,

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

Definition 1 Let X be a q-uniformly smooth Banach space and $T, A: X \to X$ be two single-valued mappings. Then T is said to be

(i) accretive if $\langle T(x) - T(y), J_q(x - y)\rangle \ge 0$, $\forall x, y \in X$;

(ii) (γ, α)-relaxed cocoercive with respect to A, if there exist constants $\gamma, \alpha > 0$ such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \ge -\gamma\|T(x) - T(y)\|^q + \alpha\|x - y\|^q, \quad \forall x, y \in X;$$

(iii) μ-Lipschitz continuous if there exists a constant $\mu > 0$ such that

$$\|T(x) - T(y)\| \le \mu\|x - y\|, \quad \forall x, y \in X.$$

Definition 2 Let $\eta: X \times X \to X$ and $g: X \to X$ be single-valued mappings, $M: X \to 2^X$ be set-valued mapping. Then

(i) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \le \tau\|x - y\|, \quad \forall x, y \in X.$$

(ii) g is said to be r-strongly η-accretive, if there exists a constant $r > 0$ such that

$$\langle g(x) - g(y), J_q(\eta(x, y))\rangle \ge r\|x - y\|^2, \quad \forall x, y \in X;$$

(iii) M is said to be η-accretive if

$$\langle u - v, J_q(\eta(x, y))\rangle \ge 0, \quad \forall x, y \in X, \ \forall u \in M(x), \ \forall v \in M(y);$$

(iv) M is said to be g-η-accretive if M is η-accretive and $(g + \lambda M)(X) = X$ for all $\lambda > 0$.

Lemma 2[6] Let $g: X \to X$ be a r-strongly η-accretive mapping, $\eta: X \times X \to X$ be τ-Lipschitz continuous and $M: X \to 2^X$ be an g-η-accretive mapping. Then the resolvent operator $R_{M,\lambda}^{\eta}: X \to X$ is $\frac{\tau}{r}$-Lipschitz continuous, i.e.,

$$\|R_{M,\lambda}^{\eta}(x) - R_{M,\lambda}^{\eta}(y)\| \le \frac{\tau}{r}\|x - y\|, \quad \forall x, y \in X.$$

Lemma 3[8] Let $\{a_n\}$, $\{b_n\}$ be two nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that

$$a_{n+1} \le (1 - t_n)a_n + b_n t_n, \quad \forall n \ge n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$. Then $a_n \to 0 (n \to \infty)$.

Let X be a real q-uniformly smooth Banach space, $S_i, T_i, g_i: X \to X$, $\eta_i: X \times X \to X$ $(i = 1, 2)$ be nonlinear single-valued mappings. Suppose that $M_i: X \to 2^X$ be g_i-η_i-accretive mappings $(i = 1, 2)$. Now we consider the following problem:

Find $x, y \in X$ such that

$$\begin{cases} 0 \in g_1(x) - g_1(y) + \rho_1(S_1(y) + T_1(y)) + \rho_1 M_1(x), \\ 0 \in g_2(y) - g_2(x) + \rho_2(S_2(x) + T_2(x)) + \rho_2 M_2(y), \end{cases} \tag{1}$$

where $\rho_i > 0 (i = 1, 2)$ are two constants. Problem (1) is called a system of g-η-accretive mapping inclusions in

Banach spaces.

For a suitable choices of the mappings $S_1, S_2, T_1, T_2, g_1, g_2, \eta_1, \eta_2, M_1, M_2$ and the spaces X, problem (1) includes many system of variational inequality (inclusion) problems as special cases, see for example[1-4] and the references therein.

2 Main Results

Lemma 4 (x, y) is a solution of problem (1) if and only if (x, y) satisfies the following relation

$$\begin{cases} x = R^{g_1}_{M_1,\rho_1}[g_1(y) - \rho_1(S_1(y) + T_1(y))], & \rho_1 > 0, \\ y = R^{g_2}_{M_2,\rho_2}[g_2(x) - \rho_2(S_2(x) + T_2(x))], & \rho_2 > 0 \end{cases} \tag{2}$$

Proof This directly follows from Definition of resolvent operator $R_{M,\rho}$.

Algorithm 1 For arbitrarily chosen initial point $x_0 \in X$, compute the sequences $\{x_n\}, \{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n R^{g_1}_{M_1,\rho_1}[g_1(y_n) - \rho_1(S_1(y_n) + T_1(y_n))], \\ y_n = (1-\beta_n)x_n + \beta_n R^{g_2}_{M_2,\rho_2}[g_2(x_n) - \rho_2(S_2(x_n) + T_2(x_n))], \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1], \forall n \geq 0$.

Theorem 1 Let X be a q-uniformly smooth Banach space. For $i = 1, 2$, let $\eta_i: X \times X \to X$ be τ_i-Lipschitz continuous mappings, $g_i: X \to X$ be r_i-strongly η_i-accretive and a_i-Lipschitz continuous mappings, $M_i: X \to 2^X$ be g_i-η_i-accretive mappings. Let $S_i: X \to X$ be b_i-Lipschitz continuous, $T_i: X \to X$ be (μ_i, γ_i)-relaxed cocoercive with respect to g_i and ξ_i-Lipschitz continuous. If there exist constants $\rho_i > 0$ such that

(i) $0 < \frac{\tau_1}{r_1}[(a_1^q - q\rho_1\gamma_1 + q\rho_1\mu_1\xi_1^q + C_q\rho_1^q\xi_1^q)^{\frac{1}{q}} + \rho_1 b_1] < 1$,

(ii) $0 < \frac{\tau_2}{r_2}(a_2^q - q\rho_2\gamma_2 + q\rho_2\mu_2\xi_2^q + C_q\rho_2^q\xi_2^q)^{\frac{1}{q}} + \rho_2 b_2 < 1$,

where C_q is the constant as in Lemma 1. Then problem (1) has a unique solution.

Proof For any given $\rho_i > 0 (i = 1, 2)$, define a mapping $F: X \to X$ as follows:

$$F(x) = R^{g_1}_{M_1,\rho_1}[g_1(y) - \rho_1(S_1(y) + T_1(y))], \tag{3}$$

where $y = R^{g_2}_{M_2,\rho_2}[g_2(x) - \rho_2(S_2(x) + T_2(x))], \quad \forall x \in X$.

For all $x_1, x_2 \in X$, it follows from (3) and Lemma 2 that

$$\begin{aligned} \|F(x_1) - F(x_2)\| &\leq \| R^{g_1}_{M_1,\rho_1}[g_1(y_1) - \rho_1(S_1(y_1) + T_1(y_1))] - \\ &\quad R^{g_1}_{M_1,\rho_1}[g_1(y_2) - \rho_1(S_1(y_2) + T_1(y_2))] \| \\ &\leq \frac{\tau_1}{r_1}(\|g_1(y_1) - g_1(y_2) - \rho_1(T_1(y_1) - T_1(y_2))\| + \rho_1\|S_1(y_1) - S_1(y_2)\|). \end{aligned} \tag{4}$$

By assumptions, we have

$$\begin{aligned} &\|g_1(y_1) - g_1(y_2) - \rho_1(T_1(y_1) - T_1(y_2))\|^q \\ &\leq \|g_1(y_1) - g_1(y_2)\|^q - q\rho_1\langle T_1(y_1) - T_1(y_2), J_q(A_1(y_1) - A_1(y_2))\rangle + C_q\rho_1^q\|T_1(y_1) - T_1(y_2)\|^q \\ &\leq (a_1^q - q\rho_1\gamma_1 + q\rho_1\mu_1\xi_1^q + C_q\rho_1^q\xi_1^q)\|y_1 - y_2\|^q, \end{aligned} \tag{5}$$

$$\|S_1(y_1) - S_1(y_2)\| \leq b_1\|y_1 - y_2\|. \tag{6}$$

Combining (4) − (6), we have

$$\|F(x_1) - F(x_2)\| \leq \frac{\tau_1}{r_1}(\theta_1 + \rho_1 b_1)\|y_1 - y_2\|, \tag{7}$$

where $\theta_1 = (a_1^q - q\rho_1\gamma_1 + q\rho_1\mu_1\xi_1^q + C_q\rho_1^q\xi_1^q)^{\frac{1}{q}}$.

Similarly, we can prove that

$$\|y_1 - y_2\| \leq \frac{\tau_2}{r_2}(\theta_2 + \rho_2 b_2)\|x_1 - x_2\|, \tag{8}$$

where $\theta_2 = (a_2^q - q\rho_2\gamma_2 + q\rho_2\mu_2\xi_2^q + C_q\rho_2^q\xi_2^q)^{\frac{1}{q}}$.

By (7) and (8), we have

$$F(x_1)-F(x_2) \quad h_1h_2 \quad x_1-x_2 \quad , \tag{9}$$

where $h_1=\frac{1}{r_1}(\ _1+\ _1b_1)$, $h_2=\frac{2}{r_2}(\ _2+\ _2b_2)$.

It follows from conditions (i) and (ii) that $0<h_i<1$ $(i=1,2)$. Thus, (9) implies that F is a contractive mapping. Hence, by the Banach contraction principle, there exists a unique x^* such that $x^*=F(x^*)$. Let

$$y^*=R^{g_2}_{M_2,\ _2}[g_2(x^*)-\ _2(S_2(x^*)+T_2(x^*))].$$

From the definition of F, we have

$$\begin{cases} x^*=R^{g_1}_{M_1,\ _1}[A_1(y^*)-\ _1(S_1(y^*)+T_1(y^*))],\\ y^*=R^{g_2}_{M_2,\ _2}[A_2(x^*)-\ _2(S_2(x^*)+T_2(x^*))].\end{cases}$$

It follows from Lemma 4 that (x^*,y^*) is the unique solution of problem (1). This completes the proof.

Theorem 2 For $i=1,2$, let $\ _i$, g_i, S_i, T_i, M_i and X be the same as in Theorem 1, and let condition (i) and (ii) and of Theorem 1 hold. Let the iterative sequences $\{x_n\}$, $\{y_n\}$ are generated by Algorithm 1 and satisfies:

$$\sum_{n=0} a_n = \quad , \quad \lim_n(1-\ _n)=0. \tag{12}$$

Then the iterative sequences $\{x_n\}$, $\{y_n\}$ converges strongly to the unique solution (x^*,y^*) of problem (1).

Proof since (x^*,y^*) be the solution of problem (1), Applying Algorithm 1 and Lemma 4, we have

$$\begin{aligned} x_{n+1}-x^* = &(1-\ _n)(x_n-x^*)+\ _n(R^{g_1}_{M_1,\ _1}[g_1(y_n)-\ _1(S_1(y_n)+T_1(y_n))]-\\ &R^{g_1}_{M_1,\ _1}[g_1(y^*)-\ _1(S_1(y^*)+T_1(y^*))])\end{aligned}$$

$$(1-\ _n)\quad x_n-x^* \quad +\ _n\frac{1}{r_1}\quad g_1(y_n)-g_1(y^*)+\ _1[(S_1(y_n)+T_1(y_n))-(S_1(y^*)+T_1(y^*))]$$

$$(1-\ _n)\quad x_n-x^* \quad +\ _n\frac{1}{r_1}(\ _1 \quad S_1(y_n)-S_1(y^*) \quad + \quad g_1(y_n)-g_1(y^*)-\ _1(T_1(y_n)-T_1(y^*)) \quad). \tag{11}$$

Since g_1 is a_1-Lipschitz continuous, T_1 be $(\ _1,\ _1)$-relaxed cocoercive with respect to g_1 and $\ _1$-Lipschitz continuous, we have

$$\begin{aligned} &g_1(y_n)-g_1(y^*)-\ _1(T_1(y_n)-T_1(y^*)) \quad ^q \quad g_1(y_n)-g_1(y^*) \quad ^q-q\ _1\ T_1(y_n)-T_1(y^*)\ ),\\ &J_q(g_1(y_n)-g_1(y^*)) \ +C_q\ ^q_1\ \ T_1(y_n)-T_1(y^*) \quad ^q \quad (a_1^q-q\ _1\ _1+q\ _1\ _1\ ^q_1+C_q\ ^q_1\ ^q_1) \quad y_n-y^* \quad ^q. \end{aligned} \tag{12}$$

Since S_1 is $\ _1$-Lipschitz continuous, we have

$$S_1(y_n)-S_1(y^*) \quad b_1 \quad y_n-y^* \quad . \tag{13}$$

By (11)-(13), we obtain

$$x_{n+1}-x^* \quad (1-\ _n)\quad x_n-x^* \quad +\ _n\frac{1}{r_1}(\ _1+\ _1b_1)\quad y_n-y^* \quad , \tag{14}$$

where $\ _1=(a_1^q-q\ _1\ _1+q\ _1\ _1\ ^q_1+C_q\ ^q_1\ ^q_1)^{\frac{1}{q}}$.

Similarly, by Algorithm 1 and condition (iii), we can prove that

$$\begin{aligned} y_n-y^* \quad &(1-\ _n)\quad x_n-y^* \quad +\ _n\frac{2}{r_2}(\ _2+\ _2b_2)\quad x_n-x^* \\ &(1-\ _n)\quad x_n-x^* \quad +\ _n\quad x_n-x^* \quad +(1-\ _n)\quad x^*-y^* \\ &\quad x_n-x^* \quad +(1-\ _n)\quad x^*-y^* \quad , \end{aligned} \tag{15}$$

where $\ _2=(a_2^q-q\ _2\ _2+q\ _2\ _2\ ^q_2+C_q\ ^q_2\ ^q_2)^{\frac{1}{q}}$.

It follows from (14) and (15) that

$$+\ x_{n+1}-x^*\ +\ [\ [1-A_n(1-h_1)]+\ x_n-x^*\ +\ +A_nh_1(1-B_n)L, \tag{16}$$

where $h_1 = \frac{S_1}{r_1}(H_1 + Q_{K_1})$.

Let $a_n = \|x_n - x^*\|$, $t_n = \alpha_n(1 - h_1)$, $b_n = \frac{h_1}{1 - h_1}(1 - \beta_n)L$. It is easy to verify that the conditions of Lemma 3 are satisfied. Thus, by (16) and Lemma 3, we have $\lim_{n\to\infty} a_n = 0$, i.e., $\|x_n - x^*\| \to 0 (n \to \infty)$. And by (15), we also obtain that $\|y_n - x^*\| \to 0 (n \to \infty)$.

This completes the proof.

Remark 1 Theorems 1 and 2 generalized and improve the corresponding results in [1-4].

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