凹角型区域上双曲型外问题的人工边界条件

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[摘要] 研究凹角型区域上双曲型外问题的人工边界条件. 利用构造法获得了圆形人工边界上精确的和近似的人工边界条件. 利用新得的人工边界条件, 用有限差分方法求解相应问题的数值解. 最后给出数值例子以示文中所得的人工边界条件的有效性.

[关键词] 凹角区域, 双曲外问题, 人工边界条件, 数值解
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Exterior Hyperbolic Problems With Concave Angles

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Abstract: In this paper, the artificial boundary condition for the exterior hyperbolic problems with concave angles and its numerical methods are studied. A circular artificial boundary Γ_R is first introduced, three kinds of equivalent exact and approximate artificial boundary conditions are obtained on circular boundary by a constructive method. Secondly, we propose new artificial boundary conditions to reduce the given problem a computational problem in a bounded domain, which is equivalent to the original problem, and the finite difference method is used to solved the reduced problem. Finally, some numerical examples are presented to demonstrate the performance of artificial boundary conditions.

Key words: concave domain, exterior hyperbolic problem, artificial boundary condition(ABC), numerical solution

科学和工程计算中的许多问题都可归结为无界区域上的边值(或初边值)问题, 其中依赖时间的发展型外问题更是有着极重要的应用[1-6]. 例如, 波的传播和扩散问题可以归结为无界区域上的波动方程[1-4]. 正是由于区域的无界性, 给数值计算或求解过程带来了一定的困难. 在过去的 20多年里, 数值求解无界区域上依赖时间问题的一种十分有效方法称之为"人工边界条件方法". 这一方法的主要思想可归结为: (i) 引入一个人工边界 B, 将原无界区域分为两个不重叠的子区域, 一个有界区域 Ω_{int} 和一个具有典型内边界的无界区域 Ω_{ext}; (ii) 通过分析无界区域上的问题, 获得人工边界 B 上精确的或近似的人工边界条件; (iii) 数值方法求解有界区域 Ω_{int} 上的问题. 在上述求解过程中, 最关键的就是 (ii). 为了保证方法的可行性和有效性, 在人工边界 B 上得到的边界条件必须是精确的或是足够精确的, 且简单易实现. 此类人工边界条件可表示为如下形式

$$\frac{\partial u}{\partial n} = \mathscr{K}_u + \mathscr{H}[f],$$

其中$\frac{\partial u}{\partial n}$是函数 u 在 $\mathscr{B}$ 上的外法向导数, $\mathscr{K}$是自然积分算子(或称为 Dirichlet-to-Neumann(DtN) 映射). 上式称之为自然积分方程(或是 DtN 边界条件).

本文采用构造法, 研究凹型区域上双曲型外问题人工边界条件及其数值计算.

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1　问题的描述

设 Ω为 $\mathbf{R}^2$上具有角度 α的凹角型区域, $0<\alpha\leqslant 2\pi$, 其边界为 Γ_n, Γ_0和 Γ_α(Γ_n为光滑的曲线段, Γ_0和 Γ_α为两条射线). 对任意固定的正实数 T, 记 $J\triangleq(0,T]$. 考虑如下初边值问题

$$\frac{\partial^2 u}{\partial t^2}=c^2\Delta u+f(\boldsymbol{x},t),\ (\boldsymbol{x},t)\in\Omega\times J, \tag{1}$$

$$u=0,\ (\boldsymbol{x},t)\in(\Gamma_0\cup\Gamma_\alpha)\times J, \tag{2}$$

$$\frac{\partial u}{\partial n}(\boldsymbol{x},t)=g_n(\boldsymbol{x},t),\ (\boldsymbol{x},t)\in\Gamma_n\times J, \tag{3}$$

$$\frac{\partial u}{\partial n}(\boldsymbol{x},0)=g(\boldsymbol{x}),\ u(\boldsymbol{x},0)=h(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega \tag{4}$$

其中, $\boldsymbol{x}\triangleq(x,y)$, Δ为二维 Laplace算子, 即 $\Delta\triangleq\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, u为波在区域 Ω中传播时的位移, c表示波在媒介中的传播速度. 为简单起见, 不妨假定 $c=1$. 函数$f(\boldsymbol{x},t)$, $g_n(\boldsymbol{x})$, $g(\boldsymbol{x})$, $h(\boldsymbol{x})$均为已知光滑函数, 且$f(\boldsymbol{x},t)$, $g(\boldsymbol{x})$, $h(\boldsymbol{x})$具有紧支集, 即存在一个半径为 R的圆 Γ_R, 当 $\|\boldsymbol{x}\|>R$时, 有

$$f(\boldsymbol{x},t)=0,\ g(\boldsymbol{x})=0,\ h(\boldsymbol{x})=0 \tag{5}$$

对于问题(1)~(5), 先引入一条人工边界, 采用构造法给出了圆形人工边界上三类等价的精确的和近似的人工边界条件. 利用所得的人工边界条件, 将原外问题归化为有界区域上的等价性问题, 用有限差分方法进行数值求解. 最后给出数值例子, 数值结果表明文中所得的人工边界条件是有效的.

2　人工边界条件

引入一条中心在坐标原点半径为R的圆弧 Γ_R为人工边界, Γ_R将 Ω分成一个有界区域 Ω_{int}和一个无界区域 Ω_{ext}, $\Omega_{\text{ext}}\triangleq\{(r,\theta)\mid 0\leqslant\theta\leqslant\alpha,\ r>R\}$. 如果能获得 Γ_R上的边界条件, 我们可以将问题(1)~(5)转化为有界区域 Ω_{int}上的问题. 为此, 我们先考虑如下的初边值问题:

$$\frac{\partial^2 u}{\partial t^2}=\Delta u,\ (x,t)\in\Omega_{\text{ext}}\times J, \tag{6}$$

$$u=0,\ (x,t)\in(\Gamma_0\cup\Gamma_\alpha)\times J,\ r>R, \tag{7}$$

$$u=u(x,t)\mid_{\Gamma_R},\ (x,t)\in\Gamma_R\times J, \tag{8}$$

$$\frac{\partial u}{\partial t}(x,0)=0,\ u(x,0)=0,\ x\in\Omega_{\text{ext}}, \tag{9}$$

$$u\to 0\ \text{当}\ \|x\|\to+\infty. \tag{10}$$

或者在极坐标的形式下, 问题(6)~(10)可表示为

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2},\ r>R,\ 0<\theta<\alpha,\ 0<t\leqslant T, \tag{6'}$$

$$u\mid_{\theta=0}=u\mid_{\theta=\alpha}=0,\ r>R,\ 0<t\leqslant T, \tag{7'}$$

$$u=u(R,\theta,t),\ 0<\theta<\alpha,\ 0<t\leqslant T, \tag{8'}$$

$$\frac{\partial u}{\partial t}(r,\theta,t)=0,\ u(r,\theta,t)=0,\ r>R,\ 0<\theta<\alpha, \tag{9'}$$

$$u\to 0\ \text{当}\ \|x\|\to+\infty. \tag{10'}$$

由于 $u(R,\theta,t)$是未知函数, 所以问题(6′)~(10′)不能独立求解. 易知函数 $u(r,\theta,t)$可表示为如下形式

$$u(r,\theta,t)=\sum_{n=1}^{+\infty}u_n(r,t)\sin\frac{n\pi}{\alpha}\theta,$$

所以, 可得

$$u(R,\theta,t)=\lim_{r\to\mathbf{R}^+}u(r,\theta,t)=\sum_{n=1}^{+\infty}u_n(r,t)\sin\frac{n\pi}{\alpha}\theta,$$

其中

$$u_n(R, t) = \frac{2}{\alpha}\int_0^{\alpha} u(R, \theta, t)\sin\frac{n\pi}{\alpha}\theta d\theta$$

经过简单的计算可得 $u_n(r, t)$, $n = 1, 2, \cdots$, 满足

$$\frac{\partial^2 u_n}{\partial t^2} = \frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r} - \frac{n^2\pi}{r^2\alpha^2}u_n,\ r > R,\ 0 < t \leqslant T, \tag{11}$$

$$u_n(r, t)|_{r=R} = u_n(R, t),\ 0 < t \leqslant T. \tag{12}$$

$$u_n(r, 0)| = 0, \frac{\partial u_n}{\partial t} = 0, r > R. \tag{13}$$

对于 (11) ~ (13) 的求解, 我们可以考虑如下问题:

$$\frac{\partial^2 G_n}{\partial t^2} = \frac{\partial^2 G_n}{\partial r^2} + \frac{1}{r}\frac{\partial G_n}{\partial r} - \frac{n^2\pi}{r^2\alpha^2}G_n,\ r > R,\ 0 < t \leqslant T, \tag{14}$$

$$\frac{\partial G_n(R, t)}{\partial t} = \frac{\pi}{\alpha}, \tag{15}$$

$$G_n(r, 0)| = 0, \frac{\partial G_n(r, 0)}{\partial t} = 0, r > R. \tag{16}$$

对任意 $\omega > 0$, 若函数 $\sin(\omega t)W(r)$ 是方程 (14) 的解, 则 $W(r)$ 应满足下面的方程

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \left(\omega^2 - \frac{n^2\pi^2}{r^2\alpha^2}\right)W = 0 \tag{17}$$

方程 (17) 是一个$\frac{n\pi}{\alpha}$阶的 Bessel方程, 有两个线性无关的解 $J_{\frac{n\pi}{\alpha}}(\omega r)$ 与 $Y_{\frac{n\pi}{\alpha}}(\omega r)$. 因此, 对任意 $\omega > 0$,

$$\frac{\sin(\omega t)}{\omega^2}\cdot\frac{J_{\frac{n\pi}{\alpha}}(\omega r)Y_{\frac{n\pi}{\alpha}}(\omega R) - J_{\frac{n\pi}{\alpha}}(\omega R)Y_{\frac{n\pi}{\alpha}}(\omega r)}{J^2_{\frac{n\pi}{\alpha}}(\omega R) + Y^2_{\frac{n\pi}{\alpha}}(\omega R)}$$

是方程 (14) 的解. 令

$$G_n^*(r, t) = \frac{2}{\alpha}\int_0^{+\infty}\frac{\sin(\omega t)}{\omega^2}\cdot\frac{J_{\frac{n\pi}{\alpha}}(\omega r)Y_{\frac{n\pi}{\alpha}}(\omega R) - J_{\frac{n\pi}{\alpha}}(\omega R)Y_{\frac{n\pi}{\alpha}}(\omega r)}{J^2_{\frac{n\pi}{\alpha}}(\omega R) + Y^2_{\frac{n\pi}{\alpha}}(\omega R)}d\omega.$$

如果 $G_n^*(r, t)$已知, 可以验证它一定是 (14) 的解. 为了分析 $G_n^*(r, t)$的性质, 应用 Bessel函数[7] 的一些结论. 当 $|z| \to +\infty$ 时,

$$J_v(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right),$$

$$Y_v(z) = \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right),$$

$$\frac{1}{J_v^2(z) + Y_v^2(z)} \sim \frac{\pi z}{2}.$$

注 **1**　(1) $G_n^*(r, t)$ 在区域 $[R, +\infty) \times [0, T]$ 上绝对可积.

(2) 经计算, 我们可以得到

$$\frac{\partial G_n^*(r, t)}{\partial r} = \frac{2}{\alpha}\int_0^{+\infty}\frac{\sin(\omega t)}{\omega^2}\cdot\frac{\frac{d}{dr}\left(J_{\frac{n\pi}{\alpha}}(\omega r)\right)Y_{\frac{n\pi}{\alpha}}(\omega R) - J_{\frac{n\pi}{\alpha}}(\omega R)\frac{d}{dr}\left(Y_{\frac{n\pi}{\alpha}}(\omega r)\right)}{J^2_{\frac{n\pi}{\alpha}}(\omega R) + Y^2_{\frac{n\pi}{\alpha}}(\omega R)}d\omega, \tag{18}$$

$$\frac{\partial G_n^*(r, t)}{\partial t} = \frac{2}{\alpha}\int_0^{+\infty}\frac{\cos(\omega t)}{\omega}\cdot\frac{J_{\frac{n\pi}{\alpha}}(\omega r)Y_{\frac{n\pi}{\alpha}}(\omega R) - J_{\frac{n\pi}{\alpha}}(\omega R)Y_{\frac{n\pi}{\alpha}}(\omega r)}{J^2_{\frac{n\pi}{\alpha}}(\omega R) + Y^2_{\frac{n\pi}{\alpha}}(\omega R)}d\omega. \tag{19}$$

(3) 在 (18) 式取 $r = R$, 并注意到 $J'_v(z)Y_v(z) - J_v(z)Y'_v(z) = -\frac{2}{\pi z}$, 可得

$$\frac{\partial G_n^*(R, t)}{\partial r} = -\frac{4}{\pi\alpha R}\int_0^{+\infty}\frac{\sin(\omega t)}{\omega^2}\cdot\frac{1}{J^2_{\frac{n\pi}{\alpha}}(\omega R) + Y^2_{\frac{n\pi}{\alpha}}(\omega R)}d\omega,\ t > 0$$

由文献 [8] 的 P679内容可知

$$\int_0^{+\infty}\frac{1}{\omega}\cdot\frac{J_\nu(\omega r)Y_\nu(\omega R)-J_\nu(\omega R)Y_\nu(\omega r)}{J_\nu^2(\omega R)+Y_\nu^2(\omega R)}\mathrm{d}\omega=-\frac{\pi}{2}\left(\frac{R}{r}\right)^\nu,\ r>R,$$

从而可得

$$\frac{\partial G_n^*(r,0)}{\partial t}=\frac{2}{\alpha}\int_0^{+\infty}\frac{1}{\omega}\cdot\frac{J_{\frac{n\pi}{\alpha}}(\omega r)Y_{\frac{n\pi}{\alpha}}(\omega R)-J_{\frac{n\pi}{\alpha}}(\omega R)Y_{\frac{n\pi}{\alpha}}(\omega r)}{J_{\frac{n\pi}{\alpha}}^2(\omega R)+Y_{\frac{n\pi}{\alpha}}^2(\omega R)}\mathrm{d}\omega=-\frac{\pi}{\alpha}\left(\frac{R}{r}\right)^{\frac{n\pi}{\alpha}},\ r>R.$$

(4) $G_n^*(r,t)$ 是方程(14)的弱解. 令

$$G_n(r,t)=\frac{\pi}{\alpha}\left(\frac{R}{r}\right)^{\frac{n\pi}{\alpha}}t+G_n^*(r,t),\tag{20}$$

可以验证, $G_n(r,t)$ 是方程(14)～(16)的解.

由 Duhame原理知, $u_n(r,t)$ 是方程(11)～(13)的解.

$$u_n(r,t)=\int_0^t\frac{\partial u_n(R,\tau)}{\partial\tau}\frac{\partial G_n(r,t-\tau)}{\partial t}\mathrm{d}\tau=\int_0^t\frac{\partial u_n(R,\tau)}{\partial\tau}\left[\frac{\partial G_n^*(r,t-\tau)}{\partial t}+\frac{\pi}{\alpha}\left(\frac{R}{r}\right)^{\frac{n\pi}{\alpha}}\right]\mathrm{d}\tau=$$

$$\frac{\pi}{\alpha}\left(\frac{R}{r}\right)^{\frac{n\pi}{\alpha}}u_n(R,t)+\int_0^t\frac{\partial^2u_n(R,\tau)}{\partial\tau^2}G_n^*(r,t-\tau)\mathrm{d}\tau,\ n=1,2,\cdots$$

因此, 有

$$\frac{\partial u_n(R,t)}{\partial r}=-\frac{n\pi^2}{\alpha^2R}u_n(R,t)+\int_0^t\frac{\partial^2u_n(R,\tau)}{\partial\tau^2}\frac{\partial G_n^*(R,t-\tau)}{\partial r}\mathrm{d}\tau.\tag{21}$$

令

$$HZ_{\frac{n\pi}{\alpha}}(t)=\frac{4}{\pi\alpha}\int_0^{+\infty}\frac{\sin(\omega t)}{\omega^2}\frac{1}{J_{\frac{n\pi}{\alpha}}^2(\omega)+Y_{\frac{n\pi}{\alpha}}^2(\omega)}\mathrm{d}\omega,$$

则有

$$\frac{\partial G_n^*(R,t)}{\partial r}=-HZ_{\frac{n\pi}{\alpha}}\left(\frac{t}{R}\right).\tag{22}$$

利用积分 $\int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x=\frac{\pi}{2}$, 函数 $HZ_{\frac{n\pi}{\alpha}}(t)$ 可以改写为

$$HZ_{\frac{n\pi}{\alpha}}(t)=\frac{4}{\pi\alpha}\int_0^{+\infty}\frac{\sin(\omega t)}{\omega^2}\left[\frac{1}{J_{\frac{n\pi}{\alpha}}^2(\omega)+Y_{\frac{n\pi}{\alpha}}^2(\omega)}-\frac{1}{2}\pi\omega\right]\mathrm{d}\omega+\frac{\pi}{\alpha},\tag{23}$$

$$HZ'_{\frac{n\pi}{\alpha}}(t)=\frac{4}{\pi\alpha}\int_0^{+\infty}\frac{\cos(\omega t)}{\omega}\left[\frac{1}{J_{\frac{n\pi}{\alpha}}^2(\omega)+Y_{\frac{n\pi}{\alpha}}^2(\omega)}-\frac{1}{2}\pi\omega\right]\mathrm{d}\omega,\tag{24}$$

$$HZ''_{\frac{n\pi}{\alpha}}(t)=-\frac{4}{\pi\alpha}\int_0^{+\infty}\sin(\omega t)\left[\frac{1}{J_{\frac{n\pi}{\alpha}}^2(\omega)+Y_{\frac{n\pi}{\alpha}}^2(\omega)}-\frac{1}{2}\pi\omega\right]\mathrm{d}\omega,\tag{25}$$

$$HZ_{\frac{n\pi}{\alpha}}(0^+)=\frac{\pi}{\alpha},\tag{26}$$

$$HZ'_{\frac{n\pi}{\alpha}}(0^+)=\frac{\pi}{2\alpha}-\frac{n\pi^2}{\alpha^2}.\tag{27}$$

这里的(27)式仍是一个猜测, 目前还难以证明, 但我们可以用数值结果进行验证. 图 1为函数 $HZ_{\frac{n\pi}{\alpha}}(t)$, $HZ'_{\frac{n\pi}{\alpha}}(t)$, 当 $n=1,2,3$, $\alpha=\frac{7\pi}{4}$ 时的图像.

将(22)代入(21), 可得

$$\frac{\partial u_n(R,t)}{\partial r}=-\frac{n\pi^2}{\alpha^2R}u_n(R,t)-\int_0^t\frac{\partial^2u_n(R,\tau)}{\partial\tau^2}HZ_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\tau=$$

$$-\frac{n\pi^2}{\alpha^2R}u_n(R,t)-\frac{\pi}{\alpha}\frac{\partial u_n(R,t)}{\partial t}-\frac{1}{R}\int_0^t\frac{\partial u_n(R,\tau)}{\partial\tau}HZ'_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\tau=$$

$$-\frac{\pi}{2\alpha R}u_n(R,t)-\frac{\pi}{\alpha}\frac{\partial u_n(R,t)}{\partial t}-\frac{1}{R^2}\int_0^tu_n(R,\tau)HZ''_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\tau.\tag{28}$$

由(28), 我们得到(6)～(10)的边界 Γ_R 的人工边界条件

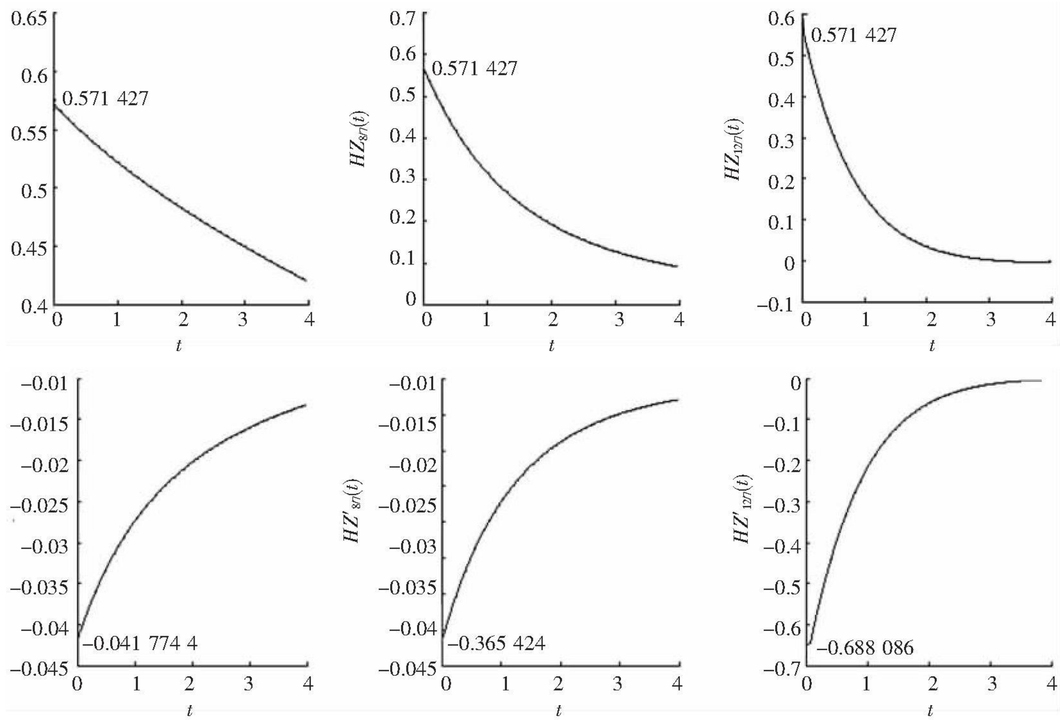

图 1　函数 $HZ_{\frac{n\pi}{\alpha}}(t)$、$HZ'_{\frac{n\pi}{\alpha}}(t)$ 的图像 ($n=1, 2, 3, \alpha=\frac{7\pi}{4}$)

Fig 1　The curve of $HZ_{\frac{n\pi}{\alpha}}(t)$、$HZ'_{\frac{n\pi}{\alpha}}(t)$ ($n=1, 2, 3, \alpha=\frac{7\pi}{4}$)

$$\frac{\partial u(R,\theta,t)}{\partial r}=-\frac{2\pi^2}{\alpha^3R}\sum_{n=1}^{+\infty}n\int_0^{\alpha}u(R,\phi,t)\cdot SS(\phi,\theta)\,\mathrm{d}\phi-\frac{2}{\alpha}\sum_{n=1}^{+\infty}\int_0^t\int_0^{\alpha}\frac{\partial^2u(R,\phi,\tau)}{\partial\tau^2}\cdot SS(\phi,\theta)\cdot HZ_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\phi\,\mathrm{d}\tau=\mathscr{K}^0_{\infty}(u(R,\theta,t)),\tag{29}$$

$$\frac{\partial u(R,\theta,t)}{\partial r}=-\frac{2\pi^2}{\alpha^3R}\sum_{n=1}^{+\infty}n\int_0^{\alpha}u(R,\phi,t)\cdot SS(\phi,\theta)\,\mathrm{d}\phi-\frac{2\pi}{\alpha^2}\sum_{n=1}^{+\infty}\int_0^{\alpha}\frac{\partial u(R,\phi,t)}{\partial t}\cdot SS(\phi,\theta)\,\mathrm{d}\phi-\frac{2}{\alpha R}\sum_{n=1}^{+\infty}\int_0^t\int_0^{\alpha}\frac{\partial u(R,\phi,\tau)}{\partial\tau}\cdot SS(\phi,\theta)\cdot HZ'_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\phi\,\mathrm{d}\tau=\mathscr{K}^1_{\infty}(u(R,\theta,t)),\tag{30}$$

$$\frac{\partial u(R,\theta,t)}{\partial r}=-\frac{\pi}{\alpha^2R}\sum_{n=1}^{+\infty}n\int_0^{\alpha}u(R,\phi,t)\cdot SS(\phi,\theta)\,\mathrm{d}\phi-\frac{2\pi}{\alpha^2}\sum_{n=1}^{+\infty}n\int_0^{\alpha}\frac{\partial u(R,\phi,t)}{\partial t}\cdot SS(\phi,\theta)\,\mathrm{d}\phi-\frac{2}{\alpha R^2}\sum_{n=1}^{+\infty}\int_0^t\int_0^{\alpha}u(R,\phi,\tau)\cdot SS(\phi,\theta)\cdot HZ''_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\phi\,\mathrm{d}\tau=\mathscr{K}^2_{\infty}(u(R,\theta,t)),\tag{31}$$

其中 $SS(\phi,\theta)\triangleq\sin\frac{n\pi}{\alpha}\phi\sin\frac{n\pi}{\alpha}\theta$. $\mathscr{K}^0_{ND}(u(R,\theta,t))$ 为 $\mathscr{K}^0_{\infty}(u(R,\theta,t))$ 取有限部分而舍去无穷部分得,

$$\mathscr{K}^0_{ND}(u(R,\theta,t))=-\frac{2\pi^2}{\alpha^3R}\sum_{n=1}^{ND}n\int_0^{\alpha}u(R,\phi,t)\cdot SS(\phi,\theta)\,\mathrm{d}\phi-\frac{2}{\alpha}\sum_{n=1}^{ND}\int_0^t\int_0^{\alpha}\frac{\partial^2u(R,\phi,\tau)}{\partial\tau^2}\cdot SS(\phi,\theta)\cdot HZ_{\frac{n\pi}{\alpha}}\left(\frac{t-\tau}{R}\right)\mathrm{d}\phi\,\mathrm{d}\tau.\tag{32}$$

同理可得 $\mathscr{K}^i_{ND}(u(R,\theta,t))$ 为 $\mathscr{K}^i_{\infty}(u(R,\theta,t))$, $i=0, 1, 2$ 取有限部分的表达式. 所以, 我们得到人工边界 Γ_R 上的近似人工边界条件

$$\frac{\partial u(R,\theta,t)}{\partial r}=\mathscr{K}^i_{ND}(u(R,\theta,t)),\ i=0,1,2;\ ND=1,2,\cdots,\tag{33}$$

利用 (33), 凹角型区域上问题 (1) ~ (5) 可以归结为如下问题:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + f(x, t), \ (x, t) \in \Omega_{\rm int} \times J, \tag{34}$$

$$u = 0, \ (x, t) \in (\Gamma_0 \cup \Gamma_\alpha) \times J, \tag{35}$$

$$\frac{\partial u}{\partial n}(x, t) = g_n(x, t), \ (x, t) \in \Gamma_n \times J, \tag{36}$$

$$\frac{\partial u(x, 0)}{\partial t} = g(x), \ u(x, 0) = h(x), \ x \in \Omega_{\rm int}, \tag{37}$$

$$\frac{\partial u(x, t)}{\partial n} = \mathcal{K}^i_{ND}(u(x, t)), \ (x, t) \in \Gamma_R \times J. \tag{38}$$

3 数值例子

为了验证文中的人工边界条件的有效性, 本节给出一些数值例子, 为简单起见, 假定$f(x, t) \equiv 0$, $c = 1$且Γ_n为中心在原点半径为1的圆弧, 即$\Gamma_n = \{(r, \theta) \mid r = 1, 0 \leqslant \theta \leqslant \alpha\}$. 引入人工边界条件$\Gamma_R$后, 我们仅需要求解如下问题:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \ 1 < r < R, \ 0 \leqslant \theta \leqslant \alpha, \ 0 < t \leqslant T, \tag{39}$$

$$u(r, 0, t) = u(r, \alpha, t) = 0, \ 1 < r < R, \ 0 < t \leqslant T, \tag{40}$$

$$\frac{\partial u(1, \theta, t)}{\partial n} = g(\theta, t), \ 0 \leqslant \theta \leqslant \alpha, \ 0 < t \leqslant T, \tag{41}$$

$$u(r, \theta, 0) = u_0(r, \theta), \ \frac{\partial u}{\partial t}(r, \theta, 0) = v_0(r, \theta), \ 1 < r < R, \ 0 \leqslant \theta \leqslant \alpha, \tag{42}$$

$$\frac{\partial u(x, t)}{\partial n} = \mathcal{K}^i_{ND}(u(x, t)), \ (x, t) \in \Gamma_R \times J. \tag{43}$$

其中$R = 4$, $T = 0.5$. 我们应用标准的二阶中心差分格式求解问题(39)~(43). 我们先将区间$[0, T]$等分K等份,

$$0 = t_1 < t_2 < \cdots < t_{K+1} = T,$$

然后, 我们沿r轴方向将区间$[1, R]$和$[0, \alpha]$分别分成M和J等份,

$$1 = r_1 < r_2 < \cdots < r_{M+1} = R, \ 0 = \theta_1 < \theta_2 < \cdots < \theta_{J+1} = \alpha.$$

设$\tau \equiv \Delta t = \frac{T}{K}$, $h \equiv \Delta r = \frac{R-1}{M}$, 及$\sigma \equiv \Delta\theta = \frac{\alpha}{J}$. 在本文中, 取$\tau = \frac{1}{2}h$. 由中心差分格式, 可得如下的计算公式

$$\frac{u^{k+1}_{m,j} - 2u^k_{m,j} + u^{k-1}_{m,j}}{\tau^2} = \frac{u^k_{m+1,j} - 2u^k_{m,j} + u^k_{m-1,j}}{h^2} + \frac{1}{r_m}\frac{u^k_{m+1,j} - u^k_{m-1,j}}{2h} + \frac{1}{r_m^2}\frac{u^k_{m,j+1} - 2u^k_{m,j} + u^k_{m,j-1}}{\sigma^2},$$

$$2 \leqslant m \leqslant M, \ 1 \leqslant k \leqslant K, \ 2 \leqslant j \leqslant J, \tag{44}$$

$$u^k_{m,1} = 0, \ u^k_{m,J+1} = 0, \ 1 \leqslant m \leqslant M+1, \ 1 \leqslant k \leqslant K+1, \tag{45}$$

$$\frac{u^1_{2,j} - u^k_{1,j}}{h} = -g(\theta_j, t_k), \ 1 \leqslant k \leqslant K+1, \ 1 \leqslant j \leqslant J+1, \tag{46}$$

$$u^1_{m,j} = u_0(r_i, \theta_j), \ 1 \leqslant m \leqslant M+1, \ 1 \leqslant j \leqslant J+1, \tag{47}$$

$$\frac{u^2_{m,j} - u^1_{m,j}}{\tau} = v_0(r_i, \theta_j), \ 1 \leqslant m \leqslant M+1, \ 1 \leqslant j \leqslant J+1, \tag{48}$$

$$\frac{u^k_{M+1,j} - u^k_{M-1,j}}{2h} = \mathcal{K}^i_{ND}(u(r^k_{M,j}, \theta^k_{M,j})), \ 1 \leqslant k \leqslant K+1, \ 2 \leqslant j \leqslant J, \ i = 0, 1, 2 \tag{49}$$

现取

$$g(1, \theta, t) = -\sum_{n=1}^{+\infty} \frac{n\pi}{\alpha} \frac{K'_{\frac{n\pi}{\alpha}}\left(\frac{n\pi}{\alpha}\right)}{K_{\frac{n\pi}{\alpha}}\left(\frac{n\pi}{\alpha}\right)} e^{\frac{-n\pi}{\alpha}t} \sin\frac{n\pi}{\alpha}\theta,$$

$$u_0(r,\theta)=\sum_{n=1}^{+\infty}\frac{K_{\frac{n\pi}{\alpha}}\left(\frac{n\pi r}{\alpha}\right)}{K_{\frac{n\pi}{\alpha}}\left(\frac{n\pi}{\alpha}\right)}\sin\frac{n\pi}{\alpha}\theta,$$

$$v_0(r,\theta)=-\sum_{n=1}^{+\infty}\frac{n\pi}{\alpha}\frac{K_{\frac{n\pi}{\alpha}}\left(\frac{n\pi r}{\alpha}\right)}{K_{\frac{n\pi}{\alpha}}\left(\frac{n\pi}{\alpha}\right)}\sin\frac{n\pi}{\alpha}\theta$$

对不同的 M, J, K, 图 2分别给出利用不同的人工边界条件时数值结果的相对误差 $E_1=\dfrac{\|u(r,\theta,t)-u_n(r,\theta,t)\|_{1,\Omega_T}}{\|u_n(r,\theta,t)\|_{1,\Omega_T}}$.

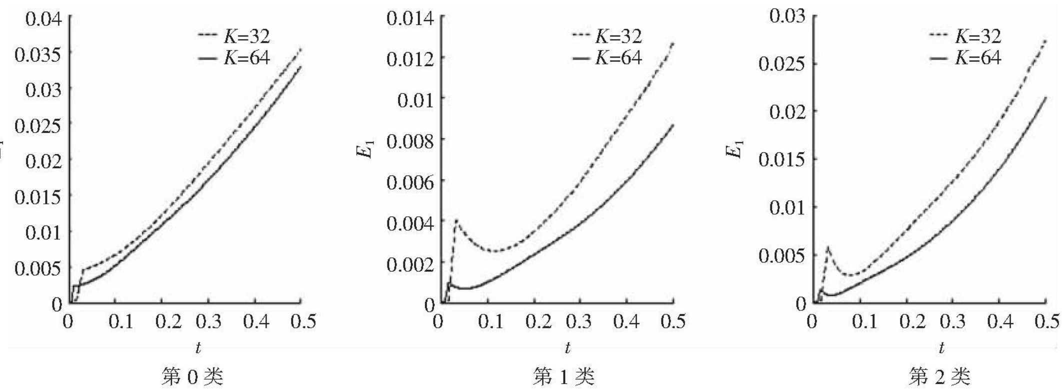

图 2　利用不同人工边界条件时的相对误差 $\dfrac{\|u(r,\theta,t)-u_n(r,\theta,t)\|_{1,\Omega_T}}{\|u_n(r,\theta,t)\|_{1,\Omega_T}}$

Fig 2　The relative error plot with different artificial boundary coditions $\dfrac{\|u(r,\theta,t)-u_n(r,\theta,t)\|_{1,\Omega_T}}{\|u_n(r,\theta,t)\|_{1,\Omega_T}}$

本文给出了二维凹型区域上双曲型初边值外问题的三类精确的和近似的人工边界条件. 在引入人工边界后, 利用所获得人工边界条件我们将原无界外区域问题化为等价的有界区域的计算问题, 从而较易进行数值求解. 从数值结果看, 文中所提出的方法对求解二维凹型区域上的双曲型初边值外问题是可行的、有效的.

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