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平面非线性系统中心焦点的待定系数判定法

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[摘要] Poincaré 所创立的形式级数法和后继函数法,是判定平面非线性系统中心焦点的经典方法,这两种方法都存在计算复杂的困难. 本文在形式级数法的基础上,利用待定系数法,建立关于形式级数各项系数的代数方程组,实现对平面系统中心焦点的判定和焦点量的计算;避开了后继函数法或形式级数法中所出现的两个无穷级数的乘积以及定积分计算问题,并举例说明此方法具有简洁和有效性.

[关键词] 平面系统,形式级数,中心,焦点

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Undetermined Coefficient Method on the Criterion of the Center-Focus for Nonlinear Planar Systems

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Abstract: The formal series and successive function created by Poincaré are two classical methods to determine the center focus for planar nonlinear systems but both methods have computational complexity. In this paper, following Poincaré's formal series method and the application of undetermined coefficient method, the algebraic equations of each coefficient of the formal series are formed, and the determination and focal value calculation of the center focus for planar system are confirmed. This method avoids the product of two infinite series arising from successive function or formal series method and the problem of the calculation of definite integral, and thus illustrates its simplicity and effectiveness.

Key words: planar system, formal series, center, fine focus

在微分方程定性理论的研究中,关于平面系统

$$\begin{cases}\dfrac{dx}{dt}=-y+X(x,y),\\ \dfrac{dy}{dt}=x+Y(x,y),\end{cases}\tag{1}$$

当 $X(x,y)$, $Y(x,y)$ 是解析函数,且 $X(0,0)=Y(0,0)=0$ 时,$O(0,0)$ 是系统(1)的中心或焦点[1-3]. 如果 $O(0,0)$ 是系统(1)的 k 阶细焦点时,对系统(1)的系数作适当的微小扰动,可在 $O(0,0)$ 的附近恰好跳出 k 个极限环. 因此,系统(1)的焦点量的计算,在研究系统(1)的极限环的个数与分布中,具有重要价值.

在系统(1)的中心焦点判定中,经典方法是 Poincaré 所创立的后继函数法和形式级数法,这两种方法都存在计算复杂的困难,至今仍是许多学者关注的问题,并相继建立各种方法. 如利用法域和特殊方向的方法[4]、积分因子法[5]、递推方法[6]、Lie 不变量[7]等方法. 当

$$X(x,y)=\sum_{k=1}^{n}P_{2k+1}(x,y),Y(x,y)=\sum_{k=1}^{n}Q_{2k+1}(x,y),$$

其中,$P_{2k+1}(x,y)$ 和 $Q_{2k+1}(x,y)$ 为 $2k+1$ 次齐次多项式,利用若干个二阶矩阵表示系统(1)解决其中心焦点的判定[8],也是一个值得关注的方法.

本文根据形式级数法,利用待定系数法,建立关于形式级数各项系数的代数方程组,实现对系统(1)

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中心焦点的判定和焦点量的计算; 避开了后继函数法或形式级数法中所出现的两个无穷级数的乘积以及定积分计算问题.

对系统(1),根据形式级数法,则有如下结果.

定理 1 $O(0,0)$ 是系统(1) 的中心的充要条件是,存在收敛级数

$$F(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=3}^{+\infty} F_i(x,y), \tag{2}$$

其中,$F_i(x,y)$ 是 x,y 的 i 次齐次多项式函数($i = 3,4,\cdots,$),使得

$$\varphi(x,y) = \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(1)} = (-y + X(x,y))\left(x + \sum_{i=3}^{+\infty}\frac{\partial F_i}{\partial x}\right) + (x + Y(x,y))\left(y + \sum_{i=3}^{+\infty}\frac{\partial F_i}{\partial y}\right), \tag{3}$$

$$\frac{\partial^k \varphi(0,0)}{\partial x^p \partial y^q} = 0, p + q = k, k = 0,1,2,\cdots \tag{4}$$

证明 将系统(1) 改写为

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y + \sum\limits_{i=2}^{+\infty} X_i(x,y) = \sum\limits_{i=1}^{+\infty} X_i(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + \sum\limits_{i=2}^{+\infty} Y_i(x,y) = \sum\limits_{i=1}^{+\infty} Y_i(x,y), \end{cases} \tag{1'}$$

其中,$X_1(x,y) = -y, Y_1(x,y) = x, X_i(x,y), Y_i(x,y)$ 是 x,y 的 i 次齐次多项式函数($i \geqslant 2$).

由形式级数法,立知必要性成立. 下证充分性. 因

$$F(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=3}^{+\infty} F_i(x,y) = \sum_{i=2}^{+\infty} F_i(x,y), F_2(x,y) = \frac{1}{2}(x^2 + y^2),$$

$$F_i(x,y) = a_{i,0}x^i + a_{i,1}x^{i-1}y + a_{i,2}x^{i-2}y^2 + \cdots + a_{i,i}y^i, i = 3,4,\cdots,$$

$$\varphi(x,y) = \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(1)} = \sum_{i+j=3}^{+\infty}\sum_{i\geqslant 1, j\geqslant 2}\left(X_i\frac{\partial F_j}{\partial x} + Y_i\frac{\partial F_j}{\partial y}\right),$$

显然,$X_i\dfrac{\partial F_j}{\partial x} + Y_i\dfrac{\partial F_j}{\partial y}$ 是 x,y 的 $i + j - 1$ 次齐次多项式. 故当 $i + j = k + 1$ 时,

$$\sum_{i\geqslant 1, j\geqslant 2, i+j=k+1}\left(X_i\frac{\partial F_j}{\partial x} + Y_i\frac{\partial F_j}{\partial y}\right) = \sum_{i=1}^{k-1}\left(X_i\frac{\partial F_{k-i+1}}{\partial x} + Y_i\frac{\partial F_{k-i+1}}{\partial y}\right) =$$

$$X_1\frac{\partial F_k}{\partial x} + Y_1\frac{\partial F_k}{\partial y} + X_2\frac{\partial F_{k-1}}{\partial x} + Y_2\frac{\partial F_{k-1}}{\partial y} + \cdots + X_{k-1}\frac{\partial F_2}{\partial x} + Y_{k-1}\frac{\partial F_2}{\partial y},$$

和

$$X_2\frac{\partial F_{k-1}}{\partial x} + Y_2\frac{\partial F_{k-1}}{\partial y} + \cdots + X_{k-1}\frac{\partial F_2}{\partial x} + Y_{k-1}\frac{\partial F_2}{\partial y}$$

都是关于 x,y 的 k 次齐次多项式. 因此$\dfrac{\partial^k\varphi(0,0)}{\partial x^p\partial y^q}, p + q = k \geqslant 3$,仅与 $\varphi(x,y)$ 的 k 次多项式有关,而与其他的各项无关.

由于

$$X_1\frac{\partial F_k}{\partial x} + Y_1\frac{\partial F_k}{\partial y} = -y\frac{\partial F_k}{\partial x} + x\frac{\partial F_k}{\partial y} =$$

$$-y(ka_{k,0}x^{k-1} + (k-1)a_{k,1}x^{k-2}y + \cdots + 2a_{k,k-2}xy^{k-2} + a_{k,k-1}y^{k-1}) +$$

$$+ x(a_{k,1}x^{k-1} + 2a_{k,2}x^{k-2}y + \cdots + (k-1)a_{k,k-1}xy^{k-2} + ka_{k,k}y^{k-1}) =$$

$$a_{k,1}x^k + (-ka_{k,0} + 2a_{k,2})x^{k-1}y + [-(k-1)a_{k,1} + 3a_{k,3}]x^{k-2}y^2 + [-(k-2)a_{k,2} + 4a_{k,4}]x^{k-3} +$$

$$+ \cdots + [-3a_{k,k-3} + (k-1)a_{k,k-1}]x^2y^{k-2} + [-2a_{k,k-2} + ka_{k,k}]xy^{k-1} - a_{k,k-1}y^k,$$

记

$$X_2\frac{\partial F_{k-1}}{\partial x} + Y_2\frac{\partial F_{k-1}}{\partial y} + \cdots + X_{k-1}\frac{\partial F_2}{\partial x} + Y_{k-1}\frac{\partial F_2}{\partial y} =$$

$$b_{k,0}x^k + b_{k,1}x^{k-1}y + b_{k,2}x^{k-2}y^2 + \cdots + b_{k,k-1}xy^{k-1} + b_{k,k}y^k, \tag{5}$$

其中系数 $b_{k,p}(p=0,1,2,\cdots,k)$ 与 $X_2,Y_2,F_2,\cdots,X_{k-1},Y_{k-1},F_{k-1}$ 的各项系数有关. 又 $X_2,Y_2,\cdots,X_{k-1},Y_{k-1}$ 的各项系数是已知的,故可记

$$b_{k,p}=b_{k,p}(a_{k-1,0},a_{k-1,1},\cdots,a_{k-1,k-1},\cdots,a_{3,0},a_{3,1},a_{3,2},a_{3,3}),p=0,1,2,\cdots,k,$$

它是关于 $a_{k-1,0},\cdots,a_{k-1,k-1},\cdots,a_{3,0},\cdots,a_{3,3}$ 的线性函数. 再由条件(4),得到

$$\begin{cases}\dfrac{\partial^k\varphi(0,0)}{\partial x^k}=k!(a_{k,1}+b_{k,0})=0,\\ \dfrac{\partial^k\varphi(0,0)}{\partial x^{k-1}\partial y}=(k-1)!(-ka_{k,0}+2a_{k,2}+b_{k,1})=0,\\ \dfrac{\partial^k\varphi(0,0)}{\partial x^{k-2}\partial y^2}=(k-2)!2![-(k-1)a_{k,1}+3a_{k,3}+b_{k,2}]=0,\\ \cdots\cdots\cdots\\ \dfrac{\partial^k\varphi(0,0)}{\partial x^2\partial y^{k-2}}=2!(k-2)![-3a_{k,k-3}+(k-1)a_{k,k-1}+b_{k,k-2}]=0,\\ \dfrac{\partial^k\varphi(0,0)}{\partial x\partial y^{k-1}}=(k-1)!(-2a_{k,k-2}+ka_{k,k}+b_{k,k-1})=0,\\ \dfrac{\partial^k\varphi(0,0)}{\partial y^k}=k!(-a_{k,k-1}+b_{k,k})=0,\end{cases}$$

故 $\varphi(x,y)$ 的 k 次多项式 $\sum\limits_{i\geqslant 1,j\geqslant 2,i+j=k+1}\left(X_i\dfrac{\partial F_j}{\partial x}+Y_i\dfrac{\partial F_j}{\partial y}\right)$ 各项系数为0. 于是

$$\varphi(x,y)=\frac{\mathrm{d}F}{\mathrm{d}t}\bigg|_{(1)}=\sum_{i+j=3}^{+\infty}\sum_{i\geqslant 1,j\geqslant 2}\left(X_i\frac{\partial F_j}{\partial x}+Y_i\frac{\partial F_j}{\partial y}\right)=0.$$

又当 $r=\sqrt{x^2+y^2}$ 充分小时,$F(x,y)$ 是正定函数,故原点为中心.

引理1　关于 $a_{k,0},a_{k,1},\cdots,a_{k,k}$ 的线性方程组(6),当 k 为奇数时,方程组(6)必有惟一解. 当方程组(6)的解不惟一或无解时,则 k 必为偶数.

$$\begin{cases}-a_{k,1}=b_{k,0},\\ ka_{k,0}-2a_{k,2}=b_{k,1},\\ (k-1)a_{k,1}-3a_{k,3}=b_{k,2},\\ \cdots\cdots\cdots\\ 3a_{k,k-3}-(k-1)a_{k,k-1}=b_{k,k-2},\\ 2a_{k,k-2}-ka_{k,k}=b_{k,k-1},\\ a_{k,k-1}=b_{k,k}.\end{cases}\tag{6}$$

证明　方程组(6)的系数行列式为

$$\begin{vmatrix}0&-1&&&&\\ k&0&-2&&0&\\ &k-1&0&\ddots&&\\ &&k-2&\ddots&-(k-1)&\\ &0&&\ddots&0&-k\\ &&&&1&0\end{vmatrix}_{(k+1)\text{阶}}=k(-1)^3\begin{vmatrix}-1&0&&&&\\ k-1&0&-3&&0&\\ &k-2&0&\ddots&&\\ &&k-3&\ddots&-(k-1)&\\ &0&&\ddots&0&-k\\ &&&&1&0\end{vmatrix}_{k\text{阶}}=$$

$$1\cdot k\begin{vmatrix}0&-3&&&\\ k-2&0&-4&&\\ &k-3&0&\ddots&\\ &&k-4&\ddots&-(k-1)&\\ &&&\ddots&0&-k\\ &&&&1&0\end{vmatrix}_{(k-1)\text{阶}}=$$

$$1\cdot 3\cdot k(k-2)\begin{vmatrix} 0 & -5 & & & & \\ k-4 & 0 & -6 & & & \\ & k-5 & 0 & \ddots & & \\ & & k-6 & \ddots & -(k-1) & \\ & & & \ddots & 0 & -k \\ & & & & 1 & 0 \end{vmatrix}_{(k-3)\text{阶}}.$$

当 $k=2l+1$ 时,则有

$$\begin{vmatrix} 0 & -1 & & & & \\ 2l+1 & 0 & -2 & & & \\ & 2l & 0 & \ddots & & \\ & & 2l-1 & \ddots & -2l & \\ & & & \ddots & 0 & -(2l+1) \\ & & & & 1 & 0 \end{vmatrix}_{(2l+2)\text{阶}} =$$

$$(2l+1)!\begin{vmatrix} 0 & -(2l+1) \\ 1 & 0 \end{vmatrix} = (2l+1)!(2l+1)\neq 0,$$

故当 k 为奇数时,方程组(6) 必有惟一解.

当 $k=2l$ 时,则有

$$\begin{vmatrix} 0 & -1 & & & & \\ 2l & 0 & -2 & & & \\ & 2l-1 & 0 & \ddots & & \\ & & 2l-2 & \ddots & -(2l-1) & \\ & & & \ddots & 0 & -2l \\ & & & & 1 & 0 \end{vmatrix}_{(2l+1)\text{阶}} = \frac{1}{2}(2l)!\begin{vmatrix} 0 & -(2l-1) & 0 \\ 2 & 0 & -2l \\ 0 & 1 & 0 \end{vmatrix} = 0,$$

故方程组(6) 的解不惟一或无解时,k 必为偶数.

引理 2 在方程组(6) 中,当 $k=2l$ 时,关于 $a_{2l,0},a_{2l,2},\cdots,a_{2l,2l}$ 的线性方程组

$$\begin{cases} 2la_{2l,0}-2a_{2l,2}=b_{2l,1}, \\ (2l-2)a_{2l,2}-4a_{2l,4}=b_{2l,3}, \\ \cdots\cdots\cdots \\ 4a_{2l,2l-4}-(2l-2)a_{2l,2l-2}=b_{2l,2l-3}, \\ 2a_{2l,2l-2}-2la_{2l,2l}=b_{2l,2l-1} \end{cases} \tag{7}$$

必有解.

证明 考虑方程组

$$\begin{cases} 2la_{2l,0}-2a_{2l,2}=b_{2l,1}, \\ (2l-2)a_{2l,2}-4a_{2l,4}=b_{2l,3}, \\ \cdots\cdots\cdots \\ 4a_{2l,2l-4}-(2l-2)a_{2l,2l-2}=b_{2l,2l-3}, \\ 2a_{2l,2l-2}=b_{2l,2l-1}+2la_{2l,2l}, \end{cases} \tag{7'}$$

它是关于 $a_{2l,0},a_{2l,2},\cdots,a_{2l-1,2l-1}$ 的非齐次线性方程组. 其系数行列式为

$$\begin{vmatrix} 2l & -2 & & & & \\ & 2l-2 & -4 & & 0 & \\ & & 2l-4 & \ddots & & \\ & & & \ddots & -(2l-4) & \\ & 0 & & & 4 & -(2l-2) \\ & & & & & 2 \end{vmatrix}_{(l+1)\text{阶}} = (2l)!! \neq 0,$$

所以,对任意的 $a_{2l,2l}$,方程组(7)必有解.

定理2 $O(0,0)$ 是系统(1) m 阶细焦点的充要条件是,存在

$$F(x,y) = \frac{1}{2}(x^2+y^2) + \sum_{i=3}^{2m} F_i(x,y), \tag{8}$$

其中,$F_i(x,y)$ 是 x,y 的 i 次齐次多项式函数($i=3,4,\cdots,2m$),使得

$$\varphi(x,y) = \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(1)} = (-y+X(x,y))\left(x+\sum_{i=3}^{2m}\frac{\partial F_i}{\partial x}\right) + (x+Y(x,y))\left(x+\sum_{i=3}^{2m}\frac{\partial F_i}{\partial y}\right), \tag{9}$$

$$\frac{\partial^k\varphi(0,0)}{\partial x^p\partial y^q} = 0, p+q=k, k=0,1,\cdots,2m-1, \frac{\partial^{2m}\varphi(0,0)}{\partial x^{2m}} = \frac{\partial^{2m}\varphi(0,0)}{\partial y^{2m}} \neq 0. \tag{10}$$

证明 系统(1)采用(1′)的形式. 当 $O(0,0)$ 是系统(1) m 阶细焦点,由形式级数法,获得正定函数

$$F(x,y) = \frac{1}{2}(x^2+y^2) + \sum_{i=3}^{2m} F_i(x,y),$$

且

$$\left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(1)} = u(x^2+y^2)^m + \Phi_{2m+1}(x,y),$$

其中 $\frac{1}{(2m)!}\frac{\partial^{2m}\varphi(0,0)}{\partial x^{2m}} = \frac{1}{(2m)!}\frac{\partial^{2m}\varphi(0,0)}{\partial y^{2m}} = u \neq 0$,$\Phi_{2m+1}(x,y)$ 是关于 x,y 的最低次幂为 $2m+1$ 的解析函数. 据此,必要性显然成立.

再证充分性. 由定理1的证明过程,立得关系式

$$\begin{cases} a_{k,1}+b_{k,0}=0, \\ -ka_{k,0}+2a_{k,2}+b_{k,1}=0, \\ -(k-1)a_{k,1}+3a_{k,3}+b_{k,2}=0, \\ \cdots\cdots\cdots \\ -3a_{k,k-3}+(k-1)a_{k,k-1}+b_{k,k-2}=0, \\ -2a_{k,k-2}+ka_{k,k}+b_{k,k-1}=0, \\ -a_{k,k-1}+b_{k,k}=0, \end{cases} \tag{6′}$$

当 $k=3,4,\cdots,2m-1$ 时成立. 当 $k=2m$ 时,由于

$$\frac{\partial^{2m}\varphi(0,0)}{\partial x^{2m}} = (2m)!(a_{2m,1}+b_{2m,0}) = \frac{\partial^{2m}\varphi(0,0)}{\partial y^{2m}} = (2m)!(-a_{2m,2m-1}+b_{2m,2m}) \neq 0, \tag{11}$$

关系式(6′)不再成立. 当 $k=2m$ 时,在关系式(6′)中,再由引理2,关于 $a_{2m,0}, a_{2m,2}, \cdots, a_{2m,2m}$ 的线性方程组有解,于是,$\varphi(x,y)$ 的 $2m$ 次齐次形式为

$$\begin{gathered}(a_{2m,1}+b_{2m,0})x^{2m} + [-(2m-1)a_{2m,1}+3a_{2m,3}+b_{2m,2}]x^{2m-2}y^2 + \\ [-(2m-3)a_{2m,3}+5a_{2m,5}+b_{2m,4}]x^{2m-4}y^4 + \cdots + \\ [-3a_{2m,2m-3}+(2m-1)a_{2m,2m-1}+b_{2m,2m-2}]x^{2m-2}y^2 + (-a_{2m,2m-1}+b_{2m,2m})y^{2m},\end{gathered}$$

令各项系数为

$$\begin{cases} a_{2m,1}+b_{2m,0}=u, \\ -(2m-1)a_{2m,1}+3a_{2m,3}+b_{2m,2}=0, \\ -(2m-3)a_{2m,3}+5a_{2m,5}+b_{2m,4}=0, \\ -(2m-5)a_{2m,5}+7a_{2m,7}+b_{2m,6}=0, \\ \cdots\cdots\cdots \\ -5a_{2m,2m-5}+(2m-3)a_{2m,2m-3}+b_{2m,2m-4}=0, \\ -3a_{2m,2m-3}+(2m-1)a_{2m,2m-1}+b_{2m,2m-2}=0, \\ -a_{2m,2m-1}+b_{2m,2m}=u, \end{cases}$$

即得到关于 $u, a_{2m,1}, a_{2m,3}, a_{2m,5}, \cdots, a_{2m,2m-3}, a_{2m,2m-1}$ 的线性方程组

$$\begin{cases} a_{2m,1}-u=-b_{2m,0}, \\ (2m-1)a_{2m,1}-3a_{2m,3}=b_{2m,2}, \\ (2m-3)a_{2m,3}-5a_{2m,5}=b_{2m,4}, \\ (2m-5)a_{2m,5}-7a_{2m,7}=b_{2m,6}, \\ \cdots\cdots\cdots \\ 5a_{2m,2m-5}-(2m-3)a_{2m,2m-3}=b_{2m,2m-4}, \\ 3a_{2m,2m-3}-(2m-1)a_{2m,2m-1}=b_{2m,2m-2}, \\ -a_{2m,2m-1}-u=-b_{2m,2m}. \end{cases} \tag{12}$$

它的系数行列式为

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 2m-1 & -3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2m-3 & -5 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -(2m-3) & 0 & 0 \\ 0 & 0 & 0 & \cdots & 3 & -(2m-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 \end{vmatrix} =$$

$$(-1)^{2m+1}\begin{vmatrix} 1 & & & & & \\ 2m-1 & -3 & & 0 & & \\ & 2m-3 & -5 & & & \\ & & \ddots & \ddots & & \\ & 0 & & \ddots & -(2m-3) & \\ & & & & 3 & -(2m-1) \end{vmatrix} +$$

$$(-1)^{m+2}\begin{vmatrix} 2m-1 & -3 & & & & \\ & 2m-5 & -5 & 0 & & \\ & & 2m-7 & \ddots & & \\ & & & \ddots & -(2m-3) & \\ & 0 & & & 3 & -(2m-1) \\ & & & & & -1 \end{vmatrix} =$$

$$(2m-1)!!(-1)^{m-1}+(2m-1)!!(-1)^{m+1}=(2m-1)!!2(-1)^{m+1}\neq 0,$$

所以,方程组(12) 有惟一解. 特别 u 是惟一. 因此,方程组

$$\begin{cases} a_{2m,1} - u = - b_{2m,0}, \\ 2ma_{2m,0} - 2a_{2m,2} = b_{2m,1}, \\ (2m-1)a_{2m,1} - 3a_{2m,3} = b_{2m,2}, \\ \cdots\cdots\cdots \\ 3a_{2m,2m-3} - (2m-1)a_{2m,2m-1} = b_{2m,2m-2}, \\ 2a_{2m,2m-2} - 2ma_{2m,2m} = b_{2m,2m-1}, \\ - a_{2m,2m-1} - u = - b_{2m,2m} \end{cases} \tag{6''}$$

有解,且

$$u = a_{2m,1} + b_{2m,0} = - a_{2m,2m-1} + b_{2m,2m}.$$

再由(11),得

$$\frac{\partial^{2m}\varphi(0,0)}{\partial x^{2m}} = (2m)!\,u = \frac{\partial^{2m}\varphi(0,0)}{\partial y^{2m}} = (2m)!\,u \neq 0.$$

故 $u \neq 0$,于是得到

$$\left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(1)} = u(x^{2m} + y^{2m}) + \Phi_{2m+1}(x,y),$$

其中,$\Phi_{2m+1}(x,y)$ 是关于 x,y 的最低次幂为 $2m+1$ 的解析函数. 由 Liapunov 稳定性判定定理,结论得证.

下举一例,说明使用待定系数方法判定系统(1)的中心焦点.

例 1 计算二次系统的焦点量[9]

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = - y - axy - y^2, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + ax^2. \end{cases} \tag{13}$$

解

$$F(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=3}^{+\infty}(a_{i,0}x^i + a_{i,1}x^{i-1}y + a_{i,2}x^{i-2}y^2 + \cdots + a_{i,i}y^i),$$

$$\varphi(x,y) = \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{(13)} = (-y - axy - y^2)\left(x + \sum_{i=3}^{+\infty}\frac{\partial F_i}{\partial x}\right) + (x + ax^2)\left(y + \sum_{i=3}^{+\infty}\frac{\partial F_i}{\partial y}\right),$$

$$\begin{cases} \dfrac{1}{3!}\dfrac{\partial^3\varphi(0,0)}{\partial x^3} = a_{3,1} = 0, \\ \dfrac{1}{2!\,1!}\dfrac{\partial^3\varphi(0,0)}{\partial x^2\partial y} = 2a_{3,2} - 3a_{3,0} = 0, \\ \dfrac{1}{1!\,2!}\dfrac{\partial^3\varphi(0,0)}{\partial x\partial y^2} = 3a_{3,3} - 2a_{3,1} - 1 = 0, \\ \dfrac{1}{3!}\dfrac{\partial^3\varphi(0,0)}{\partial y^3} = - a_{3,2} = 0, \end{cases}$$

解得 $(a_{3,0}, a_{3,1}, a_{3,2}, a_{3,3}) = \left(0,0,0,\frac{1}{3}\right)$.

$$\begin{cases} \dfrac{1}{4!}\dfrac{\partial^4\varphi(0,0)}{\partial x^4} - u_1 = a_{4,1} + a\,a_{3,1} - u_1 = 0, \\ \dfrac{1}{3!\,1!}\dfrac{\partial^4\varphi(0,0)}{\partial x^3\partial y} = 2a_{4,2} - 4a_{4,0} - 3a\,a_{3,0} + 2a\,a_{3,2} = 0, \\ \dfrac{1}{2!\,2!}\dfrac{\partial^4\varphi(0,0)}{\partial x^2\partial y^2} = 3a_{4,3} - 3a_{4,1} - 3a_{3,0} - 2a\,a_{3,1} + 3a\,a_{3,3} = 0, \\ \dfrac{1}{1!\,3!}\dfrac{\partial^4\varphi(0,0)}{\partial x\partial y^3} = 4a_{4,4} - 2a_{4,2} - 2a_{3,1} - a\,a_{3,2} = 0, \\ \dfrac{1}{4!}\dfrac{\partial^4\varphi(0,0)}{\partial y^4} - u_1 = - a_{4,3} - a_{3,2} - u_1 = 0, \end{cases}$$

得$(a_{4,0}, a_{4,1}, a_{4,2}, a_{4,3}, a_{4,4}, u) = \left(a_{4,4}, \frac{a}{6}, 2a_{4,4}, -\frac{a}{6}, a_{4,4}, \frac{a}{6}\right)$，其中，$a_{4,4}$ 是自由量，表明 $F(x,y)$ 是不惟一的，此时

$$F(x,y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{3}y^3,$$

$$\varphi(x,y) = \frac{\mathrm{d}F}{\mathrm{d}t}\bigg|_{(13)} = \frac{a}{6}(x^4 + y^4) + \frac{a}{6}[ax(x^4 + y^4) - 6ax^3y^2 - 3x^2y^3 + y^5] - 4a_{4,4}xy^2(x^2 + y^2).$$

当 $a > 0$ 时，$(0,0)$ 是系统(13)的一阶不稳定细焦点；当 $a < 0$ 时，$(0,0)$ 是系统(13)的一阶稳定细焦点.

当 $a = 0$ 时，取 $a_{2k,2k} = 0, k = 2,3,\cdots$，由

$$\frac{1}{q!(p-q)!}\frac{\partial^p\varphi(0,0)}{\partial x^q\partial y^{p-q}} = 0,\ p = 4, 5, \cdots, q = 0, 1, \cdots, p,$$

得到的方程组均有零解，故得到

$$F(x,y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{3}y^3.$$

易知，它是系统(13)的首次积分，且在$(0,0)$的充分小的邻域内，是正定的，故$(0,0)$是系统(13)的中心.

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