第 34 卷第 4 期 南京师大学报(自然科学版) Vol. 34 No. 4
2011 年 12 月 JOURNAL OF NANJING NORMAL UNIVERSITY(Natural Science Edition) Dec, 2011

A Multi-Symplectic Integration Method for the Poisson Equation

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Abstract: In this paper, we analyze the multi-symplectic structure and the relevant conservation laws for the Poisson equation. A multi-symplectic pseudo-spectral scheme of the Poisson equation is derived and some numerical results are presented. The high accuracy of the new-derived scheme implies that the multi-symplectic methods provide a new useful tool to study the Poisson equation.

Key words: multi-symplectic, Fourier pseudo-spectral method, Poisson equation

CLC number: O241 **Document code:** A **Article ID:** 1001-4616(2011)04-0009-04

泊松方程的一个多辛积分方法

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[摘要] 分析了泊松方程的多辛结构，推导了泊松方程的多辛拟谱格式，并得出相关守恒律，最后进行了数值试验. 数值模拟的高精度说明多辛方法为泊松方程的研究提供了一个有效的新工具.

[关键词] 多辛，Fourier 拟谱方法，泊松方程

A great deal of numerical methods have been used to solve the Poisson equation. Sun[1] proposed a 4th-order accurate scheme combining the predictive and the corrective solutions. Ma[2] presented a pseudo-spectral method which arranged the collocation points and the matrix of derivative for solving Poisson equation in polar coordinate system. Liao[3] analyzed the efficiency of Jacobi iterative parallel algorithm for solving 2D Poisson equation, and presented the design of successive over relaxation (SOR) iterative parallel algorithm. Poisson's equation was transformed to a Helmholtz-like equation which was solved iteratively, as a linear system of equations, by introducing a grid and employing fast Fourier transform by Ida M B Nielsen in Ref. [4]. John P Boyd[5] compared seven different strategies for computing spectrally-accurate approximations or differential equation solutions in a disk and have laid out the merits and flaws of each spectral option. All those algorithms, as well as other traditional algorithms, are considered from the convergence, accuracy and stability of numerical solutions. Here in this paper, we consider algorithms in a new framework of geometrical structure.

Received date: 2011-09-12.

Foundation item: Supported by the National Basic Research Program of China(2010AA012304), the National Natural Science Foundation of China (10971102 & 10871099), the Foundation for the Authors of the National Excellent Doctoral Thesis Award of China(200720) and "333 Project" Foundation of Jiangsu Province of China.

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1 Poisson Equation Structure Analysis

Consider the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y). \tag{1}$$

Let

$$\frac{\partial u}{\partial y}=v,\ \frac{\partial u}{\partial x}=w,$$

then

$$\begin{cases}\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial x}=f(x,y),\\ -\dfrac{\partial u}{\partial y}=-v,\\ -\dfrac{\partial u}{\partial x}=-w,\end{cases}$$

i. e.,

$$\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial w}{\partial y}\end{pmatrix}+\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\begin{pmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial x}\\ \dfrac{\partial w}{\partial x}\end{pmatrix}=\begin{pmatrix}f(x,y)\\-v\\-w\end{pmatrix}. \tag{2}$$

The Poisson equation can be rewritten as the following multi-symplectic form PDEs.

$$\boldsymbol{M}\boldsymbol{z}_y+\boldsymbol{K}\boldsymbol{z}_x=\nabla_z S(z), \tag{3}$$

where

$$\boldsymbol{M}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad \boldsymbol{K}=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad z=\begin{pmatrix}u\\v\\w\end{pmatrix}$$

and $S(z)=f(x,y)u-\frac{1}{2}v^2-\frac{1}{2}w^2$.

2 Full-Discrete Multi-Symplectic Conservation Law Deduction

We approximate $z(x,y)$ by $I_N z(x,y)$. The approximation $I_N z(x,y)$ has the form:

$$I_N z(x,y)=\sum_{l=0}^{M-1}\sum_{k=0}^{N-1} z_k^l\cdot g_k(x)\cdot g_l(y).$$

A numerical discretization of (3) can be written schematically as:

$$\boldsymbol{M}\sum_{l=0}^{M-1}\sum_{k=0}^{N-1} z_k^l\cdot g_k(x)\cdot\frac{\partial g_l(y)}{\partial y}+\boldsymbol{K}\sum_{l=0}^{M-1}\sum_{k=0}^{N-1} z_k^l\cdot g_l(y)\cdot\frac{\partial g_k(x)}{\partial x}=\nabla_z S(z), \tag{4}$$

$$\boldsymbol{M}\sum_{l=0}^{M-1}\sum_{k=0}^{N-1} z_k^l\, g_k(x)\,\frac{\partial g_l(y)}{\partial y}\Big|_{\substack{y=y_i\\x=x_j}}+\boldsymbol{K}\sum_{l=0}^{M-1}\sum_{k=0}^{N-1} z_k^l\, g_l(y)\,\frac{\partial g_k(x)}{\partial x}\Big|_{\substack{y=y_i\\x=x_j}}=\nabla_z S(z)\Big|_{\substack{y=y_i\\x=x_j}},$$

$$\boldsymbol{M}\sum_{l=0}^{M-1} z_j^l\frac{\partial g_l(y)}{\partial y}\Big|_{y=y_i}+\boldsymbol{K}\sum_{k=0}^{N-1} z_k^i\frac{\partial g_k(x)}{\partial x}\Big|_{x=x_j}=\nabla_z S(z_j^i). \tag{5}$$

Let

$$(\overline{\boldsymbol{D}}_1)_{il}=\frac{\partial g_l(y)}{\partial y}\Big|_{y=y_i},\quad (\boldsymbol{D}_1)_{jk}=\frac{\partial g_k(x)}{\partial x}\Big|_{x=x_j},$$

then

$$\boldsymbol{M}\sum_{l=0}^{M-1}(\overline{\boldsymbol{D}}_1)_{il}z_j^l+\boldsymbol{K}\sum_{k=0}^{N-1}(\boldsymbol{D}_1)_{jk}z_k^i=\nabla_z S(z_j^i). \tag{6}$$

Theorem

The discrete scheme has MN full-discrete multi-symplectic conservation law,

$$\sum_{l=0}^{M-1}(\bar{\boldsymbol{D}}_1)_{il}\omega_{il}+\sum_{k=0}^{N-1}(\boldsymbol{D}_1)_{jk}\kappa_{jk}=0,\begin{array}{l} i=0,1,\cdots,M-1\\ j=0,1,\cdots,N-1\end{array}, \tag{7}$$

where

$$\omega_{il}=\frac{1}{2}\left[\mathrm{d}z_j^i\wedge\boldsymbol{M}\mathrm{d}z_j^l+\mathrm{d}z_j^l\wedge\boldsymbol{M}\mathrm{d}z_j^i\right],$$

$$\kappa_{jk}=\frac{1}{2}\left[\mathrm{d}z_j^i\wedge\boldsymbol{K}\mathrm{d}z_k^i+\mathrm{d}z_k^i\wedge\boldsymbol{K}\mathrm{d}z_j^i\right].$$

Proof The variational equation associated with (6) is

$$\boldsymbol{M}\sum_{l=0}^{M-1}(\bar{\boldsymbol{D}}_1)_{il}\mathrm{d}z_j^l+\boldsymbol{K}\sum_{k=0}^{N-1}(\boldsymbol{D}_1)_{jk}\mathrm{d}z_k^i=S_{zz}(z_j^i)\,\mathrm{d}z_j^i. \tag{8}$$

Taking the wedge product with dz_j^i, then

$$\mathrm{d}z_j^i\wedge\boldsymbol{M}\sum_{l=0}^{M-1}(\bar{\boldsymbol{D}}_1)_{il}\mathrm{d}z_j^l+\mathrm{d}z_j^i\wedge\boldsymbol{K}\sum_{k=0}^{N-1}(\boldsymbol{D}_1)_{jk}\mathrm{d}z_k^i=\mathrm{d}z_j^i\wedge S_{zz}(z_j^i)\,\mathrm{d}z_j^i. \tag{9}$$

Noting the fact $\mathrm{d}z_j^i\wedge S_{zz}(z_j^i)\,\mathrm{d}z_j^i=0$, then

$$\mathrm{d}z_j^i\wedge\boldsymbol{M}\sum_{l=0}^{M-1}(\bar{\boldsymbol{D}}_1)_{il}z_j^l+\mathrm{d}z_j^i\wedge\boldsymbol{K}\sum_{k=0}^{N-1}(\boldsymbol{D}_1)_{jk}\mathrm{d}z_k^i=0. \tag{10}$$

Now we obtain the MN full-discrete multi-symplectic conservation laws:

$$\sum_{l=0}^{M-1}(\bar{\boldsymbol{D}}_1)_{il}\omega_{il}+\sum_{k=0}^{N-1}(\boldsymbol{D}_1)_{jk}\kappa_{jk}=0,\begin{array}{l} i=0,1,\cdots,M-1\\ j=0,1,\cdots,N-1\end{array},$$

where

$$\omega_{il}=\frac{1}{2}\left[\mathrm{d}z_j^i\wedge\boldsymbol{M}\mathrm{d}z_j^l+\mathrm{d}z_j^l\wedge\boldsymbol{M}\mathrm{d}z_j^i\right],$$

$$\kappa_{jk}=\frac{1}{2}\left[\mathrm{d}z_j^i\wedge\boldsymbol{K}\mathrm{d}z_k^i+\mathrm{d}z_k^i\wedge\boldsymbol{K}\mathrm{d}z_j^i\right].$$

3 Numerical Experiments

For the numerical experiments, we consider Poisson equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\sin(x)\sin(y), \tag{11}$$

with the boundary condition $u(0,y)=u(2\pi,y)=0, u(x,0)=u(x,2\pi)=0$.

According to analyze, we can obtain the full-discrete multi-symplectic equation, i.e.,

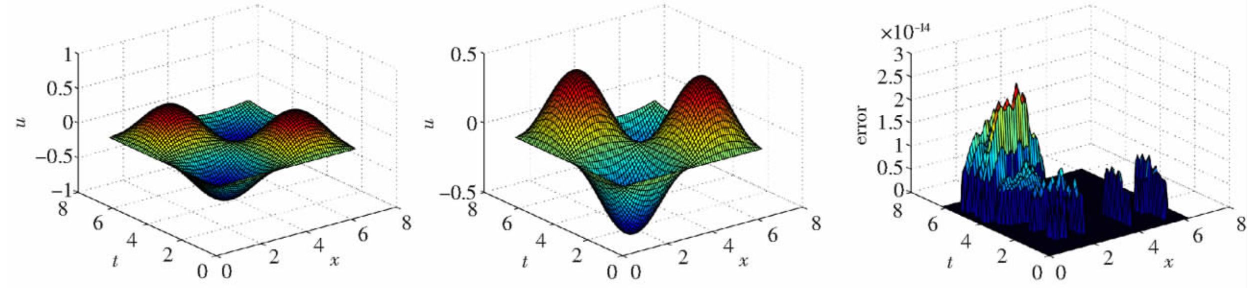
$$\sum_{l=0}^{N-1}(\bar{\boldsymbol{D}}_2)_{il}u_j^l+\sum_{k=0}^{N-1}(\boldsymbol{D}_2)_{jk}u_k^i=\sin(x_j)\sin(y_i), \tag{12}$$

where

$$(\bar{\boldsymbol{D}}_2)_{il}=\begin{cases}\dfrac{1}{2}\mu_2^2(-1)^{i+l+1}\dfrac{1}{\sin^2\left[\mu_2\dfrac{y_i-y_l}{2}\right]}, & i\neq l,\\ -\mu_2^2\dfrac{2(M/2)^2+1}{6}, & i=l,\end{cases}\qquad y_i=\frac{2\pi}{M}i,$$

$$(\boldsymbol{D}_2)_{jk}=\begin{cases}\dfrac{1}{2}\mu_1^2(-1)^{j+k+1}\dfrac{1}{\sin^2\left[\mu_1\dfrac{x_j-x_k}{2}\right]}, & j\neq k,\\ -\mu_1^2\dfrac{2(N/2)^2+1}{6}, & j=k,\end{cases}\qquad x_j=\frac{2\pi}{N}j.$$

Then we obtain Fig. 1 as follows:



(a) Numerical solutions (b) Exact solutions (c) Numerical solutions error

Fig.1 Numerical solutions

4 Conclusions

Based on the multi-symplectic structure, we construct a new method for the Poisson equation. The high accuracy of the numerical results illustrates that multi-symplectic method even structure-preserving algorithms can be applied successfully to elliptic equations.

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