

Independent Number and Degree Condition for Fractional ID- $[a, b]$ -Factor-Critical Graphs

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Abstract: A graph G is fractional independent-set-deletable $[a, b]$ -factor-critical if $G-I$ has a fractional $[a, b]$ -factor for every independent set I of G . In this paper, we prove that if $\alpha(G) \leq \frac{4b(\delta(G)-b+1)}{(a+1)^2+4b}$, then G is fractional ID- $[a, b]$ -factor-critical.

Key words: independent number, minimum degree, fractional $[a, b]$ -factor, fractional ID- $[a, b]$ -factor-critical

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分数 ID- $[a, b]$ -因子临界图的最小度与独立数条件

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[摘要] 对图 G 的每个独立集 I , 若 $G-I$ 有分数 $[a, b]$ -因子, 则 G 是分数 ID- $[a, b]$ -因子临界图. 本文证明了若 $\alpha(G) \leq \frac{4b(\delta(G)-b+1)}{(a+1)^2+4b}$, 则 G 是分数 ID- $[a, b]$ -因子临界图.

[关键词] 独立数, 最小度, 分数 $[a, b]$ -因子, 分数 ID- $[a, b]$ -因子临界图

In this paper, we only consider finite undirected graphs without loops or multiple edges. Let $G=(V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G . We write $N_G[x]$ for $N_G(x) \cup \{x\}$ and $\delta(G)$ the minimum degree of G . For $S \subseteq V(G)$, the induced subgraph of G by S is denoted by $G[S]$ and $G-S = G[V(G) \setminus S]$. A vertex set $S \subseteq V(G)$ is called independent if no two elements of S are adjacent in G . Let a and b be two integers with $1 \leq a \leq b$. An $[a, b]$ -factor of a graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for any $x \in V(G)$. Moreover, an $[a, b]$ -factor is called a k -factor if $a = b = k$. Other notations and terminologies are the same as those in [1].

Let $h: E(G) \rightarrow [0, 1]$ be a function. If $a \leq \sum_{x \in e} h(e) \leq b$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional $[a, b]$ -factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. If $a = b = k$, then a fractional $[a, b]$ -factor is called a fractional k -factor. A graph G is fractional independent-set-deletable k -factor-critical (in short, fractional ID- k -factor-critical) if $G-I$ has a fractional k -factor for every independent set I of G . A graph G is fractional independent-set-deletable $[a, b]$ -factor-critical (in short, fractional ID- $[a, b]$ -factor-critical) if $G-I$ has a fractional $[a, b]$ -factor for every independent set I of G .

Many authors have investigated $[a, b]$ -factors^[2-6] and fractional factors^[7-10]. And the following results are already known.

Theorem 1^[11] Let G be a graph, and k be an integer with $k \geq 1$. If

$$\alpha(G) \leq \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},$$

then G is fractional ID- k -factor-critical.

Theorem 2^[12] Let k be a positive integer and G be a graph of order n with $n \geq 6k - 8$. If $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID- k -factor-critical.

Theorem 3^[13] Let G be a graph of order n , and a, b be two integers with $1 \leq a \leq b$. If $n \geq \frac{(a+2b)(a+b-2)+1}{b}$ and $\delta(G) \geq \frac{(a+b)n}{a+2b}$, then G is fractional ID- $[a, b]$ -factor-critical.

Throughout the paper, we mainly prove the following theorem about independent number and minimum degree condition for fractional ID- $[a, b]$ -factor-critical graph, which is an extension of Theorem 1 in a certain sense.

Theorem 4 Let G be a graph, and let a and b be two integers with $1 \leq a \leq b$. If $\alpha(G) \leq \frac{4b(\delta(G) - b + 1)}{(a+1)^2 + 4b}$,

then G is fractional ID- $[a, b]$ -factor-critical.

1 The Proof of Theorem 4

Now we prove Theorem 4. The following lemma is very useful to our proof.

Lemma 1^[14] Let G be a graph and a, b be nonnegative integers with $a \leq b$. Then G has a fractional $[a, b]$ -factor if and only if for every subset S of $V(G)$,

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

Proof Clearly, Theorem 4 holds when $a+b=2$ (i. e., $a=b=1$). In the following we may assume that $a+b \geq 3$. Let I be an independent set of G and $H = G - I$. Obviously, $|V(H)| = n - |I|$, $\delta(G) \leq n - |I|$ and $\delta(G) \leq \delta(H) + |I|$.

By the definition of fractional ID- $[a, b]$ -factor-critical, we need only to complete the proof that H has a fractional $[a, b]$ -factor. By contradiction, we suppose that H has no fractional $[a, b]$ -factor. Then, in view of Lemma 1, there exists a subset S of $V(H)$ such that

$$\delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \leq -1, \quad (1)$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\}$. In addition, we choose subsets S and T such that $|T|$ is minimum with respect to (1).

Now we have the following claims.

Claim 1 $T \neq \emptyset$.

Otherwise, $T = \emptyset$. Then, according to (1) we obtain that

$$-1 \geq \delta_H(S, T) = b|S| \geq 0,$$

a contradiction. Hence, $T \neq \emptyset$.

Claim 2 $d_{H-S}(x) \leq a - 1$ for any $x \in T$.

In fact, if $d_{H-S}(x) \geq a$ for some $x \in T$, then,

$$\begin{aligned} -1 &\geq b|S| + d_{H-S}(T) - a|T| = b|S| + d_{H-S}(T \setminus \{x\}) + d_{H-S}(x) - a(|T| - 1 + 1) \\ &= b|S| + d_{H-S}(T \setminus \{x\}) - a|T \setminus \{x\}| = \delta_H(S, T \setminus \{x\}), \end{aligned}$$

contradicting to the choice of S and T . Therefore, Claim 2 is true.

Define

$$h = \min \{d_{H-S}(x) : x \in T\}.$$

In terms of Claim 2, we have $0 \leq h \leq a - 1$.

Claim 3 $|S| \geq \delta(G) - |I| - h$.

We choose $x_1 \in T$ such that $d_{H-S}(x_1) = h$. Thus, we obtain

$$\delta(G) \leq d_G(x_1) \leq d_{H-S}(x_1) + |I| + |S| = h + |I| + |S|,$$

which implies

$$|S| \geq \delta(G) - |I| - h.$$

This completes the proof of Claim 3.

Now we proceed to prove Theorem 4. Since $T \neq \emptyset$, in the following we shall construct a sequence of x_1, x_2, \dots, x_k of vertices of T . We take $x_1 \in T$ such that x_1 is the vertex with the least degree in $G[T]$. Let $T_1 = G[T]$ and $N_1 = N_{T_1}[x_1]$. For $i \geq 2$, if $T - \bigcup_{1 \leq j < i} N_j \neq \emptyset$, let $T_i = G[T] - \bigcup_{1 \leq j < i} N_j$. Then we choose $x_i \in T_i$ such that x_i is the vertex with the least degree in T_i , and set $N_i = N_G[x_i] \cap T_i$. Moreover, we denote the order of N_i by n_i . Continue these procedures until we reach the situation in which $T_i = \emptyset$ for some i , say for $i = k+1$. Then from the above definition we know that $\{x_1, x_2, \dots, x_k\}$ is an independent set of G . For $T \neq \emptyset$, we get $k \geq 1$.

According to the definition of N_i , we can obtain the following properties.

$$\alpha(G[T]) \geq k, \quad (2)$$

$$|T| = \sum_{1 \leq i \leq k} n_i. \quad (3)$$

As our choice of x_i implies that all vertices in N_i have degree at least $n_i - 1$ in T_i , we have

$$\sum_{1 \leq i \leq k} \left(\sum_{x \in N_i} d_{T_i}(x) \right) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i). \quad (4)$$

It is easy to see that

$$d_{H-S}(T) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i) + \sum_{1 \leq i < j \leq k} e_G(N_i, N_j) \geq \sum_{1 \leq i \leq k} (n_i^2 - n_i). \quad (5)$$

By (2) and the obvious inequality $\alpha(G) \geq \alpha(G[T])$, we have

$$\alpha(G) \geq k. \quad (6)$$

Adding $\alpha(G) \geq |I|$, the inequalities (1), Claim 3, (3), (6), $n_i - (a+1)n_i \geq -\frac{(a+1)^2}{4}$ and the assumption

$\alpha(G) \leq \frac{4b(\delta(G) - b + 1)}{(a+1)^2 + 4b}$, we get

$$\begin{aligned} -1 &\geq \delta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \geq b(\delta(G) - |I| - h) + \sum_{1 \leq i \leq k} (n_i^2 - n_i) - a \sum_{1 \leq i \leq k} n_i = \\ &b(\delta(G) - |I| - h) + \sum_{1 \leq i \leq k} (n_i^2 - (a+1)n_i) \geq b(\delta(G) - |I| - h) - \frac{(a+1)^2}{4}k \geq b(\delta(G) - h) - \\ &\frac{(a+1)^2 + 4b}{4} \cdot \alpha(G) \geq b(\delta(G) - h) - \frac{(a+1)^2 + 4b}{4} \cdot \frac{4b(\delta(G) - b + 1)}{(a+1)^2 + 4b} = b(b - h - 1) \geq 0. \end{aligned}$$

That is a contradiction. So we conclude that H has a fractional $[a, b]$ -factor. Furthermore, G is fractional ID- $[a, b]$ -factor-critical.

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