The Stability Analysis of an SEIRS Model

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Abstract:In this paper,nonlinear incidence with a more general form is considered in an SEIRS epidemic model. The model without time delay in the removed class is compared with the model with time delay in the removed class. The result shows that the dynamic behaviors of the model with time delay are different from those of the model without delay. For the model without time delay,the disease free equilibrium(DFE)is globally asymptotically stable when the basic reproduction number is smaller than one. When the basic reproduction number is bigger than one,regardless of the time delay length there exists a unique endemic equilibrium which is locally asymptotically stable under a condition. As for the model with time delay,the stability of the DFE depends on the time delay besides the basic reproduction number. Furthermore,the stability of the unique endemic equilibrium can be obtained under some conditions depending on the time delay. In addition,by numerical simulations,periodic solutions can be found from the endemic equilibrium when the time delay is in some regions.

Key words:SEIRS model,nonlinear incidence,stability,vertical transmission,time delay

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一类 SEIRS 模型稳定性分析

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[**摘要**] 建立了一个 SEIRS 流行病模型,考虑更一般形式的非线性发生率. 对恢复类中有时滞和没有时滞的模型进行了比较. 结果显示,带有时滞的模型的动力学行为与不带时滞的模型的动力学行为是不同的. 对于不带时滞的模型,如果基本再生数小于 1,无病平衡点(DFE)是全局渐近稳定的. 当基本再生数大于 1 时,不论免疫期的长短系统都存在唯一的地方病平衡点,并且在一定的条件下是局部渐近稳定的. 对于带有时滞的模型,DFE 的稳定性依赖于基本再生数和时滞. 而且,唯一的地方病平衡点的稳定性也依赖于时滞. 另外,通过数值模拟显示,当时滞在一定的范围内时,周期解有可能会出现.

[**关键词**] SEIRS 模型,非线性发生率,稳定性,垂直传播,时滞

In many epidemic models the total constant population is divided into four classes:susceptible S,exposed E,infectious I,removed R. These models are called SEIRS since the susceptible become exposed,then infectious,then removed and then susceptible again after the temporary immunity is lost. This SEIRS type models have been studied many times in previous investigations[1-16]. For example,an SEIR model incorporating density dependence in the death rate is studied in reference[11]. Then Grenhalgh studied Hopf bifurcations in a new SEIRS model with density dependent contact rate and death rate[12]. Reference[13,14]investigated the global dynamics of the SEIR

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models with a nonlinear incidence rate and with saturating contact rate, respectively. Reference [15] studied the global dynamics of the SEIR model with a bilinear incidence rate and vertical transmission. Recently, Reference [16] considered the global dynamics of the SEIR model with a standard incidence rate.

In fact, the incidence of a disease is the number of new cases per unit time and plays an important role in the study of mathematical epidemiology. The general form of incidence is written as $\beta U(N)SI$, where N is the population size depending on environment, S and I are the numbers of susceptible and infective individual at time t, respectively, β is the probability per unit time of transmitting the infection between two individuals taking part in a contact. $U(N)$ is usually called the contact rate, and $\beta U(N)$ is called adequate contact rate. In many literatures, the adequate contact rate frequently takes two forms. One is linearly proportional to the total population size N or βN, so that the corresponding incidence is bilinear form βNSI, i. e. βSI. The other is a constant k, the corresponding incidence kSI is called standard incidence rate. When the total population size N is not too large, since the number of contacts made by an infectious per unit time should increase as the total population size N increases, the linear adequate contact rate βN would be suitable. But when the total population size is quite large, since the number of contacts made by an infectious per unit time should be limited, or should grow less rapidly as the total population size N increases, the linear adequate contact rate βN is not suitable and the constant adequate contact rate k may be more realistic. Hence, the two adequate contact rates mentioned above are actually two extreme cases for the total population size N being very small and very large, respectively[8]. There also exists the incidence forming $\beta I^p S^q$ in some models[17]. They formed the values of p and q can influence the number of equilibrium and the dynamical behavior. In addition, a model with a general nonlinear incidence $\beta g(I)S$ is studied in reference[1]. It is found that multiple equilibria exist for some parameter values and periodic solutions can arise by Hopf bifurcation from the larger endemic equilibrium.

In real life, many infectious in nature transmit through both horizontal and vertical modes, such as herpes, rubella, hepatitis B, and AIDS[17-24]. Vertical transmission of diseases is the passing of an infection to offspring of infected parents. This mode of transmission plays an important role in the spread of diseases. In recent years, the studies of epidemic models incorporating vertical transmission have become one of the important areas in the mathematical theory of epidemiology[18-21] and they have largely been inspired by the works of Busenberg and Cooke[22,23].

There are many papers considering time delays in SEIRS models[1,3,25]. In reference[25], two time delays are introduced and studied in an SEIRS model, where the two delays represent the latent and immune periods, respectively. They presented local stability analysis of equilibria and obtained sufficient conditions for global stability of disease free equilibrium. By neglecting disease-related death rates in the SEIRS model in reference [25], Wang[3] shown the stability of equilibria and the uniformly persistence in the population. In reference[1] the one time delay is the immune period of the removed class and they found that the time delay can lead to periodic solutions. However, those delayed models don't present the difference between the model without time delays and the model with time delays. In this paper, we establish a new SEIRS epidemic model including a general incidence forming $\beta h(S)I$, vertical transmission and one time delay in the removed class. Furthermore, we compare the model without time delay with the model with time delay. Our goal is to determine the impact of time delay on dynamics of the SEIRS model.

1 The Model

The mathematical model described here considers four classes: $S(t)$, $E(t)$, $I(t)$, $R(t)$ denoting, respectively, the densities of the population that are susceptible, exposed (not yet infectious), infectious and recovered with temporary immunity. Obviously,

$$S(t)+E(t)+I(t)+R(t)=1.$$

In this article we consider vertical transmission. In addition, it is assumed that the nonlinear incidence is to be of the form $\beta h(S)I$, where $h(0)=0$, $h'(S)>0$ for $S\in(0,1]$. The classical bilinear incidence (mass action) has $h(S)=S$, and β is then the constant contact rate. The parameter $\eta>0$ is the constant rate for loss of immunity, $\gamma>0$ is the constant rate for recovery, $\varepsilon>0$ is the constant rate for the exposed population becoming infectious, $1-p$ is the fraction of offspring born of the infectious classes I and entered into the exposed class E, $\mu>0$ is the constant birth (and death) rate and $\tau\geqslant 0$ is the immunity period of the recovered.

Then we have the model in the following:

$$\begin{cases}\dfrac{dS}{dt}=\mu-\mu S(t)-(1-p)\mu I(t)-\beta h(S(t))I+\eta e^{-\mu\tau}R(t-\tau),\\ \dfrac{dE}{dt}=\beta h(S(t))I-(\varepsilon+\mu)E(t)+(1-p)\mu I(t),\\ \dfrac{dI}{dt}=\varepsilon E(t)-(\mu+\gamma)I(t),\\ \dfrac{dR}{dt}=\gamma I(t)-\mu R(t)-\eta e^{-\mu\tau}R(t-\tau).\end{cases}\tag{1}$$

Using the standard method, it is easy to see that the disease free equilibrium $E_0=(1,0,0,0)$ always exists. Denote $S_0=1$.

Define the basic reproductive number

$$R_0=\frac{\varepsilon((1-p)\mu+\beta h(s_0))}{(\varepsilon+\mu)(\mu+\gamma)}.\tag{2}$$

These quantities have a clear biological interpretation. Consider the case when an infectious is introduced into a purely susceptible population with size $S_0=1$. The number of susceptible that will become exposed per unit of time is $\beta h(S_0)$ from contact with the infectious and $(1-p)\mu$ from vertical transmission. $\varepsilon((1-p)\mu+\beta h(S_0))$ is the number of new infectious population. $\dfrac{1}{(\varepsilon+\mu)(\mu+\gamma)}$ is the mean infective period of the disease. Thus, R_0 gives the total number of offspring of the infectious during its life time in susceptible populations. The following section shows that the basic reproductive number R_0 provides a threshold condition for parasite extinction.

Let $(S(t),E(t),I(t),R(t))$ be any solution of (1) with non-negative initial conditions, we know that it remains non-negative anytime. And because $S(t)+E(t)+I(t)+R(t)=1$, we need only consider (1) within the region Ω, where

$$\Omega=\{(S,E,I,R)\mid S\geqslant 0,E\geqslant 0,I\geqslant 0,R\geqslant 0,S+E+I+R\leqslant 1\}.$$

Lemma 1 There exist at most two equilibria in Ω.

(i) If $R_0\leqslant 1$, system (1) has a unique DFE E_0.

(ii) If $R_0>1$, system (1) has two equilibria, the DFE E_0 and the unique endemic equilibrium $E=(S^*,E^*,I^*,R^*)$ regardless of the time delay length.

Proof From the second and third equations of Eq. (1), let their right hand equal to zero, we have

$$\beta h(S)-\frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon}+(1-p)\mu=0.\tag{3}$$

Denote the left hand side of Eq. (3) as $F(S)$. It is easy to see that

$$F(0)=(1-p)\mu-\frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon}=-p\mu-\gamma-\frac{\mu(\mu+\gamma)}{\varepsilon}<0,$$

$$F(S_0)=\beta h(S_0)+(1-p)\mu-\frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon}=\frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon}(R_0-1),$$

$$F'(S)=\beta h'(S)>0.$$

If $R_0>1$, $F(S_0)>0$ and then Eq. (3) has a unique root $S^*>0$. Hence, if $R_0>1$ system (1) has a unique endemic equilibrium $E=(S^*,E^*,I^*,R^*)$ regardless of the time delay length, where

$$I^*=\frac{1-S^*}{\frac{\mu+\gamma}{\varepsilon}+1+\frac{\gamma}{\mu+\eta e^{-\mu\tau}}},\quad E^*=\frac{\mu+\gamma}{\varepsilon}I^*,\quad R^*=\frac{\gamma}{\mu+\eta e^{-\mu\tau}}I^*.$$

Next we discuss the stabilities of E_0 and E in system(1) with $\tau=0$.

2 Stability Analysis of Equilibria With $\tau=0$

In this section, we will analyze the stability of the two equilibria of the model(1) with $\tau=0$.

Theorem 1 The disease free equilibrium E_0 of the system(1) with $\tau=0$ is globally asymptotically stable if $R_0<1$.

Proof First we show the disease free equilibrium E_0 of the system(1) with $\tau=0$ is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$. Linearizing the system(1) with $\tau=0$ around E_0, we can obtain the characteristic roots are $\lambda=-\mu,-(\mu+\eta)$ and roots of the following equation:

$$\lambda^2+(\varepsilon+2\mu+\gamma)\lambda+(\varepsilon+\mu)(\mu+\gamma)-\varepsilon((1-p)\mu+\beta h(S_0))=0. \tag{4}$$

Since

$$-(\varepsilon+2\mu+\gamma)<0,$$
$$(\varepsilon+\mu)(\mu+\gamma)-\varepsilon((1-p)\mu+\beta h(S_0))=(\varepsilon+\mu)(\mu+\gamma)(1-R_0)>0,$$

if and only if $R_0<1$, the real parts of all eigenvalues of E_0 are negative. Thus, the disease free equilibrium E_0 of the system(1) with $\tau=0$ is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$.

Next we prove that the global stability of the disease free equilibrium E_0 if $R_0<1$.

Define a function

$$V(t)=E(t)+\frac{\varepsilon+\mu}{\varepsilon}I(t).$$

The derivative of $V(t)$ along solutions of Eq. (1) with $\tau=0$ is

$$V'(t)=E'(t)+\frac{\varepsilon+\mu}{\varepsilon}I'(t)=\frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon}\left(\frac{\varepsilon((1-p)\mu+\beta h(S))}{(\varepsilon+\mu)(\mu+\gamma)}-1\right)I(t).$$

From $h'(S)>0$ and $0<S\leqslant S_0$, we have $h(S)\leqslant h(S_0)$ and then

$$V'(t)\leqslant\frac{(\varepsilon+\mu)(\mu+\gamma)}{\varepsilon}I(t)(R_0-1)\leqslant 0.$$

Since $R_0<1$, $V'(t)=0$ holds only when $I(t)=0$. It follows from Eq. (1) with $\tau=0$ that $E(t)\to 0$, $I(t)\to 0$, $R(t)\to 0$ and $S(t)\to 1$, and then $E_0=(1,0,0)$ is the largest invariant subset in the set where $V'(t)=0$. Hence, the disease free equilibrium E_0 of the system(1) with $\tau=0$ is globally asymptotically stable if $R_0<1$.

Now, we turn to consider the stability analysis of the unique endemic equilibrium E.

$$(\mathrm{H}):\beta h^l(S^*)I^*\geqslant -p\mu+\beta h(S^*).$$

Theorem 2 If $R_0>1$ and (H) is satisfied, the endemic equilibrium E of the system(1) with $\tau=0$ is locally asymptotically stable.

Proof The characteristic equation of E is

$$\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4=b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4, \tag{5}$$

where

$a_1=\varepsilon+4\mu+\gamma+\beta h^l(S^*)I^*>0$,

$a_2=\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(3\mu+\gamma)+\mu(\varepsilon+2\mu+\gamma)+(\mu+\gamma)(\varepsilon+\mu)$,

$a_3=2\mu\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(3\mu^2+2\mu\gamma+\varepsilon\gamma)+\mu(\mu+\gamma)(\varepsilon+\mu)$,

$a_4=\mu^2\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(\mu^3+\mu^2\gamma+\mu\varepsilon\gamma)$,

$b_1=-\eta<0$,

$b_2=-\eta(\mu+\beta h^l(S^*)I^*+\varepsilon+2\mu+\gamma)<0$,

$b_3=-\eta(\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\gamma)(\varepsilon+\mu))$,

$b_4=-\eta(\mu\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(\mu^2+\mu\gamma)+\mu\varepsilon\gamma)$.

If (H) is satisfied, it is easy to see $a_2>0, a_3>0, a_4>0, b_3<0, b_4<0$. Thus, the left-hand side in Eq. (5) is positive while the right-hand side is negative for all $\lambda\geqslant 0$. Then Eq. (5) does not have non-negative real solutions. Now we consider whether Eq. (5) have imaginary solutions with non-negative real parts or not.

Suppose $\lambda=u+iv$ ($u\geqslant 0$ and $v\neq 0$) is a root of Eq. (5). Then we have

$$(u+iv)^4+(a_1-b_1)(u+iv)^3+(a_2-b_2)(u+iv)^2+(a_3-b_3)(u+iv)+(a^4-b^4)=0. \tag{6}$$

Denote $A_1=a_1-b_1, A_2=a_2-b_2, A_3=a_3-b_3$ and $A_4=a_4-b_4$. Then we can obtain $A_1>0, A_2>0, A_3>0, A_4>0$ and

$$(u^2-v^2)^2-4u^2v^2+A_1u(u^2-3v^2)+A_2(u^2-v^2)+A_3u+A_4=0, \tag{7}$$

$$4uv(u^2-v^2)+A_1v(3u^2-v^2)+2A_2uv+A_3\mathrm{v}=0. \tag{8}$$

Let $u^2-v^2=m$. From (8), we obtain

$$m=-\frac{2A_1u^2+2A_2u+A_3}{4u+A_1}. \tag{9}$$

Combining (7) with (9), we can get

$$-\frac{64u^6+96A_1u^5+(48A_1^2+32A_2)u^4+(8A_1^3+32A_1A_2)u^3+4Ru^2+2QA_1u+P}{(4u+A_1)^2}=0, \tag{10}$$

where

$$P=A_1A_2A_3-A_1^2A_4-A_3^2,\ Q=A_1A_3+A_2^2-4A_4,\ R=2\ A_1^2A_2+A_1A_3+A_2^2-4A_4.$$

Denote $\beta h^l(S^*)I^*=h$ and $\beta h(S^*)=q$, then the condition $\beta h^l(S^*)I^*\geqslant -p\mu+\beta h(S^*)$ is equivalent to $h+p\mu\geqslant q$. For simplifying calculation, we substituting $q=h+p\mu-l$ ($l\geqslant 0$) into P, Q, R and we can obtain the following:

$$\begin{aligned}P=&(\varepsilon\gamma h+\varepsilon^2l+2\mu\varepsilon l+\gamma\varepsilon l+h\varepsilon l+8\mu^3+6\varepsilon\mu^2+8\mu^2\gamma+8h\mu^2+8\eta\mu^2+5\mu\varepsilon\gamma+3h\varepsilon\mu+6h\mu\gamma+6\eta\varepsilon\mu+6\eta\mu\gamma+2\eta\varepsilon h+6\eta\mu h+\\&2\eta\gamma h+2\eta\varepsilon\gamma+\varepsilon^2\eta+\varepsilon\eta^2+2\mu h^2+2\mu\eta^2+h^2\eta+\eta^2h+\varepsilon^2\gamma+\varepsilon^2\mu+\varepsilon\gamma^2+2\mu\gamma^2+\gamma^2\eta+\gamma\eta^2+\gamma^2h+\gamma h^2)(2\mu\varepsilon l+\eta\varepsilon l+\\&8\mu^3+4\eta\mu^2+4\mu^2\gamma+2\eta\mu\gamma+2\varepsilon\mu^2+\eta\varepsilon\mu+2\mu\varepsilon\gamma+\eta\varepsilon\gamma+4h\mu^2+2\eta\mu h+2h\mu\gamma+\eta\gamma h+\varepsilon\gamma h)>0,\end{aligned}$$

$$\begin{aligned}Q=&8h\mu\varepsilon l+8\gamma\mu\varepsilon l+6\mu\varepsilon^2l+16\varepsilon\mu^2l+8\eta\mu\varepsilon l+2\varepsilon^2l\gamma+2\varepsilon l\gamma h+\varepsilon^2l^2+3\varepsilon^2l\eta+\eta^2\varepsilon l+3\varepsilon l\eta\gamma+3\varepsilon l\eta h+14\varepsilon\mu\gamma h+13\varepsilon\mu h\eta+\\&21\varepsilon\mu\gamma\eta+7\varepsilon\gamma h\eta+24\mu\gamma h\eta+5\varepsilon^2\mu^2+32\varepsilon\mu^3+\varepsilon^2\gamma^2+\varepsilon^2\eta^2+48\mu^3\gamma+48\mu^3h+48\mu^3\eta+48\mu^4+6\varepsilon^2\mu\gamma+\varepsilon^2\gamma h+5\varepsilon^2\mu\eta+\\&3\varepsilon^2\gamma\eta+32\varepsilon\mu^2\gamma+16\varepsilon\mu^2h+32\varepsilon\mu^2\eta+8\varepsilon\gamma^2\mu+3\varepsilon\gamma^2h+3\varepsilon\gamma^2\eta+\varepsilon h^2\gamma+7\varepsilon\eta^2\mu+2\varepsilon\eta^2h+3\varepsilon\eta^2\gamma+40\mu^2\gamma h+40\mu^2h\eta+\\&40\mu^2\gamma\eta+8\mu\gamma^2h+8\mu\gamma^2\eta+8\mu h^2\gamma+8\mu h^2\eta+8\mu\eta^2h+8\mu\eta^2\gamma+3\gamma^2h\eta+3\gamma h^2\eta+3\gamma\eta^2h+12\mu^2\gamma^2+12\mu^2h^2+12\mu^2\eta^2+\\&\gamma^2h^2+\gamma^2\eta^2+h^2\eta^2>0,\end{aligned}$$

$$\begin{aligned}R=&24h\mu\varepsilon l+24\gamma\mu\varepsilon l+22\mu\varepsilon^2l+48\varepsilon\mu^2l+24\eta\mu\varepsilon l+6\varepsilon^2l\gamma+6\varepsilon l\gamma h+4\varepsilon^2hl+2h^2\varepsilon l+2\gamma^2\varepsilon l+2\varepsilon^3l+\varepsilon^2l^2+7\varepsilon^2l\eta+3\eta^2\varepsilon l+\\&7\varepsilon l\eta\gamma+7\varepsilon l\eta h+78\varepsilon\mu\gamma h+77\varepsilon\mu h\eta+101\varepsilon\mu\gamma\eta+27\varepsilon\gamma h\eta+108\mu\gamma h\eta+4\varepsilon^3\mu+49\varepsilon^2\mu^2+2\varepsilon^3\gamma+2\varepsilon^3\eta+192\varepsilon\mu^3+\\&5\varepsilon^2\gamma^2+5\varepsilon^2\eta^2+240\mu^3\gamma+240\mu^3h+240\mu^3\eta+240\mu^4+36\varepsilon^2\mu\gamma+14\varepsilon^2\mu h+7\varepsilon^2\gamma h+35\varepsilon^2\mu\eta+6\varepsilon^2h\eta+13\varepsilon^2\gamma\eta+\\&168\varepsilon\mu^2\gamma+120\varepsilon\mu^2h+168\varepsilon\mu^2\eta+40\varepsilon\gamma^2\mu+11\varepsilon\gamma^2h+13\varepsilon\gamma^2\eta+16\varepsilon h^2\mu+7\varepsilon h^2\gamma+6\varepsilon h^2\eta+39\varepsilon\eta^2\mu+10\varepsilon\eta^2h+\\&13\varepsilon\eta^2\gamma+192\mu^2\gamma h+192\mu^2h\eta+192\mu^2\gamma\eta+42\mu\gamma^2h+42\mu\gamma^2\eta+42\mu h^2\gamma+42\mu h^2\eta+42\mu\eta^2h+42\mu\eta^2\gamma+13\gamma^2h\eta+\\&13\gamma h^2\eta+13\gamma\eta^2h+72\mu^2\gamma^2+72\mu^2h^2+72\mu^2\eta^2+6\gamma^3\mu+2\gamma^3h+2\gamma^3\varepsilon+2\gamma^3\eta+5\gamma^2h^2+5\gamma^2\eta^2+6h^3\mu+2h^3\gamma+2h^3\eta+\\&5h^2\eta^2+6\eta^3\mu+2\eta^3h+2\eta^3\gamma+2\eta^3\varepsilon>0.\end{aligned}$$

It is easy to see the left-hand side of Eq. (10) is negative for all $u\geqslant 0$ under the condition $h+p\mu\geqslant q$. Consequently, the left-hand of Eq. (10) does not equivalent to zero, which implies Eq. (5) does not have imaginary solutions with non-negative real parts. Hence, if $R_0>1$ and (H) is satisfied, the endemic equilibrium E of the system (1) with $\tau=0$ is locally asymptotically stable.

Note that when $\tau=0$, (H) is equivalent to

$$\beta h'(S^*)(1-S^*)\geqslant\left(\frac{\mu+\gamma}{\varepsilon}+1+\frac{\gamma}{\mu+\eta}\right)\left(\frac{\mu^2+\mu\gamma+\varepsilon\gamma}{\varepsilon}\right).$$

3 Stability Analysis of Equilibria With $\tau>0$

In this section, we will turn to consider the stabilities of the two equilibria of the model (1) with $\tau>0$.

Theorem 3 If $R_0<1$ and $\mu\geqslant\eta e^{-\mu\tau}$, the DFE E_0 of the system (1) with $\tau>0$ is locally asymptotically stable.

Proof Linearizing the system(1) with $\tau>0$ around E_0 we can obtain the characteristic roots are $\lambda=-\mu$ and roots of the following equation:

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=\mathrm{e}^{-\lambda\tau}(b_1\lambda^2+b_2\lambda+b_3),\tag{11}$$

where

$a_1=\varepsilon+3\mu+\gamma>0$,

$a_2=\mu(\varepsilon+2\mu+\gamma)+(\mu+\gamma)(\varepsilon+\mu)-\varepsilon((1-p)\mu+\beta h(S_0))=\mu(\varepsilon+2\mu+\gamma)+(\mu+\gamma)(\varepsilon+\mu)(1-R_0)$,

$a_3=\mu((\mu+\gamma)(\varepsilon+\mu)-\varepsilon((1-p)\mu+\beta h(S_0)))=\mu(\mu+\gamma)(\varepsilon+\mu)(1-R_0)$,

$b_1=-\eta\mathrm{e}^{-\lambda\tau}<0$,

$b_2=-\eta\mathrm{e}^{-\lambda\tau}(\varepsilon+2\mu+\gamma)<0$,

$b_3=-\eta\mathrm{e}^{-\lambda\tau}((\mu+\gamma)(\varepsilon+\mu)-\varepsilon((1-p)\mu+\beta h(S_0)))=-\eta\mathrm{e}^{-\lambda\tau}(\mu+\gamma)(\varepsilon+\mu)(1-R_0)$.

If $R_0<1$, it is easy to see $a_2>0$, $a_3>0$, $b_2<0$, $b_3<0$. Thus, the left-hand side in Eq. (11) is positive while the right-hand side is negative for all $\lambda\geqslant 0$ and $\tau>0$. Then Eq. (11) does not have non-negative real solutions. Now we consider whether Eq. (11) have purely imaginary solutions or not.

Suppose $\lambda=\omega\mathrm{i}(\omega>0)$ is a root of Eq. (11). Then we have

$$-\omega^3\mathrm{i}-a_1\omega^2+a_2\omega\mathrm{i}+a_3=(-b_1\omega^2+b_2\omega^2\mathrm{i}+b_3)(\cos(\omega\tau)-\mathrm{i}\sin(\omega\tau)).\tag{12}$$

Separating the real and imaginary parts, we have the following system

$$-\omega^3+a_2\omega=\cos(\omega\tau)b_2\omega-\sin(\omega\tau)(b_3-b_1\omega^2),\tag{13}$$

$$-a_1\omega^2+a_3=\cos(\omega\tau)(b_3-b_1\omega^2)+\sin(\omega\tau)b_2\omega.\tag{14}$$

To eliminate the trigonometric functions we square both sides of each equation above and we add the squared equation (13) and (14) to obtain the following equation:

$$(-\omega^3+a_2\omega)^2+(-a_1\omega^2+a_3)^2=(b_2\omega)^2+(b_3-b_1\omega^2)^2,\tag{15}$$

i. e.

$$\omega^6+(a_1^2-2a_2-b_1^2)\omega^4+(a_2^2-2a_1a_3-b_2^2+2b_1b_3)\omega^2+(a_3^2-b_3^2)=0.\tag{16}$$

Let $z=\omega^2$, we obtain

$$z^3+c_1z^2+c_2z+c_3=0,\tag{17}$$

where

$c_1=a^2-2a_2-b^2=(\varepsilon+\mu)^2+(\mu+\gamma)^2+2R_0(\varepsilon+\mu)(\mu+\gamma)+(\mu^2-\eta^2\mathrm{e}^{-2\mu\tau})$,

$c_2=a^2-2a_1a_3-b_2+2b_1b_3=(\mu^2-\eta^2\mathrm{e}^{-2\mu\tau})[(\varepsilon+\mu)^2+(\mu+\gamma)^2+2R_0(\varepsilon+\mu)(\mu+\gamma)]+(\varepsilon+\mu)^2(\mu+\gamma)^2(1-R_0)^2$,

$c_3=a^2-b^2=(\varepsilon+\mu)^2(\mu+\gamma)^2(1-R_0)^2(\mu^2-\eta^2\mathrm{e}^{-2\mu\tau})$.

If $R_0<1$ and $\mu\geqslant\eta\mathrm{e}^{-\mu\tau}$, $c_1>0$, $c_2>0$ and $c_3>0$, which implies Eq. (17) does not have positive solutions and then Eq. (11) does not have purely imaginary solutions. Hence, If $R_0<1$ and $\mu\geqslant\eta\mathrm{e}^{-\mu\tau}$, the DFE E_0 of the system (1) with $\tau>0$ is locally asymptotically stable.

Now, we turn to the study of the stability of the endemic equilibrium of the model (1) with $\tau>0$.

Lemma 2 For the function $f(x)=x^4+d_1x^3+d_2x^2+d_3x+d_4$, if $d_1>0$, $d_2>0$, $d_3>0$, and $d_4>0$, then $f(x)=0$ has no positive real roots.

Proof Taking the derivative of $f(x)$ with respect to x, we obtain

$$f'(x)=4x^3+3d_1x^2+2d_2x+d_3.$$

Notice that for $x\geqslant 0$ the derivative $f'(x)>0$ and then the function $f(x)$ is an increasing function of $x\geqslant 0$. Since $f(0)=d_4>0$, $f(x)=0$ has no positive real roots.

Theorem 4 If $R_0>1$, $\mu\geqslant\eta\mathrm{e}^{-\mu\tau}$ and (H) is satisfied, the endemic equilibrium E of the system (1) with $\tau>0$ is locally asymptotically stable.

Proof The characteristic equation of E is

$$\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4=\mathrm{e}^{-\lambda\tau}(b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4),\tag{18}$$

where

$a_1=\varepsilon+4\mu+\gamma+\beta h^l(S^*)I^*>0$,

$a_2=\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(3\mu+\gamma)+\mu(\varepsilon+2\mu+\gamma)+(\mu+\gamma)(\varepsilon+\mu)$,

$a_3=2\mu\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(3\mu^2+2\mu\gamma+\varepsilon\gamma)+\mu(\mu+\gamma)(\varepsilon+\mu)$,

$a_4=\mu^2\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(\mu^3+\mu^2\gamma+\mu\varepsilon\gamma)$,

$b_1=-\eta e^{-\lambda\tau}<0$,

$b_2=-\eta e^{-\lambda\tau}(\mu+\beta h^l(S^*)I^*+\varepsilon+2\mu+\gamma)<0$,

$b_3=-\eta e^{-\lambda\tau}(\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\gamma)(\varepsilon+\mu))$,

$b_4=-\eta e^{-\lambda\tau}(\mu\varepsilon(\mu+\beta h^l(S^*)I^*((1-p)\mu+\beta h(S^*)))+(\mu+\beta h^l(S^*)I^*)(\mu^2+\mu\gamma)+\mu\varepsilon\gamma)$.

If (H) is satisfied, it is easy to see $a_2>0$, $a_3>0$, $a_4>0$, $b_3<0$, $b_4<0$. Thus, the left-hand side in Eq. (18) is positive while the right-hand side is negative for all $\lambda\geqslant 0$. Then Eq. (18) does not have non-negative real solutions. Now we consider whether Eq. (18) have purely imaginary solutions or not.

Suppose $\lambda=\omega i(\omega>0)$ is a root of Eq. (18). Then we have

$$\omega^4-a_1\omega^3 i-a_2\omega^2+a_3\omega i+a_4=(-b_1\omega^3 i-b_2\omega^2+b_3\omega i+b_4)(\cos(\omega\tau)-i\sin(\omega\tau)). \tag{19}$$

Thus, ω must satisfy the following system

$$\omega^4-a_2\omega^2+a_4=\cos(\omega\tau)(b_4-b_2\omega^2)+\sin(\omega\tau)(b_3\omega-b_1\omega^3), \tag{20}$$

$$-a_1\omega^3+a_3\omega=\cos(\omega\tau)(b_3\omega-b_1\omega_3)-\sin(\omega\tau)(b_4-b_2\omega^2). \tag{21}$$

From (20) and (21), we can get

$$(\omega^4-a_2\omega^2+a_4)^2+(-a_1\omega^3+a_3\omega)^2=(b_4-b_2\omega^2)^2+(b_3\omega-b_1\omega^3)^2, \tag{22}$$

i. e.

$$\omega^8+c_1\omega^6+c_2\omega^4+c_3\omega^2+c_4=0, \tag{23}$$

where

$c_1=a_1^2-2a_2-b_1^2$,

$c_2=a_2^2+2a_4-2a_1a_3-b_2^2+2b_1b_3$,

$c_3=a_3^2-2a_2a_4+2b_2b_4-b_3^2$,

$c_4=a_4^2-b_4^2$.

Let $z=\omega^2$ again, we obtain

$$z^4+c_1z^3+c_2z^2+c_3z+c_4=0. \tag{24}$$

By complex calculations, we can get $c_1>0$, $c_2>0$, $c_3>0$ and $c_4>0$ if $\mu\geqslant\eta e^{-\mu\tau}$ and (H) is satisfied. According to Lemma 2, Eq. (24) does not have positive roots and then Eq. (18) cannot have purely imaginary solutions. Hence, if $R_0>1$, $\mu\geqslant\eta e^{-\mu\tau}$ and (H) is satisfied, the endemic equilibrium E of the system (1) with $\tau>0$ is locally asymptotically stable.

Notice that when $\tau>0$, $\mu\geqslant\eta e^{-\mu\tau}$ and (H) is equivalent to

$$\frac{1}{\mu}\ln\frac{\eta}{\mu}\leqslant\tau\leqslant\frac{1}{\mu}\ln\frac{\eta\varepsilon^2\beta h'(S^*)(1-S^*)-(\mu+\gamma+\varepsilon)(\mu^2+\mu\gamma+\varepsilon\gamma)\eta}{(\mu+\gamma)(\mu^2+\mu\gamma+\varepsilon\gamma)-\mu\varepsilon^2\beta h'(S^*)(1-S^*)}.$$

Here, to make sense of the inequality above, following conditions are satisfied:

$$\mu<\eta,$$

and

$$\frac{(\mu+\gamma)(\mu+\varepsilon)}{\mu\varepsilon^2}(\mu^2+\mu\gamma+\varepsilon\gamma)>\beta h'(S^*)(1-S^*)\geqslant\frac{(\mu+\gamma)(\mu+\varepsilon)+\mu(\mu+\gamma+\varepsilon)}{2\mu\varepsilon^2}(\mu^2+\mu\gamma+\varepsilon\gamma).$$

Remark 1 If $\mu\geqslant\eta$, the condition $\mu\geqslant\eta e^{-\mu\tau}$ is satisfied for all nonnegative τ. However, sometimes μ is smaller than η in realistic situation. In order to satisfy the condition $\mu\geqslant\eta e^{-\mu\tau}$, the delay τ must be controlled in some regions. In fact $\mu\geqslant\eta e^{-\mu\tau}$ is equivalent to $\tau\geqslant\frac{1}{\mu}\ln\frac{\eta}{\mu}\triangleq\tau_0$. If $0<\tau<\frac{1}{\mu}\ln\frac{\eta}{\mu}$, there may be other phenomena in this system. We take some numerical simulations and get oscillations with some parameters in the following simulations.

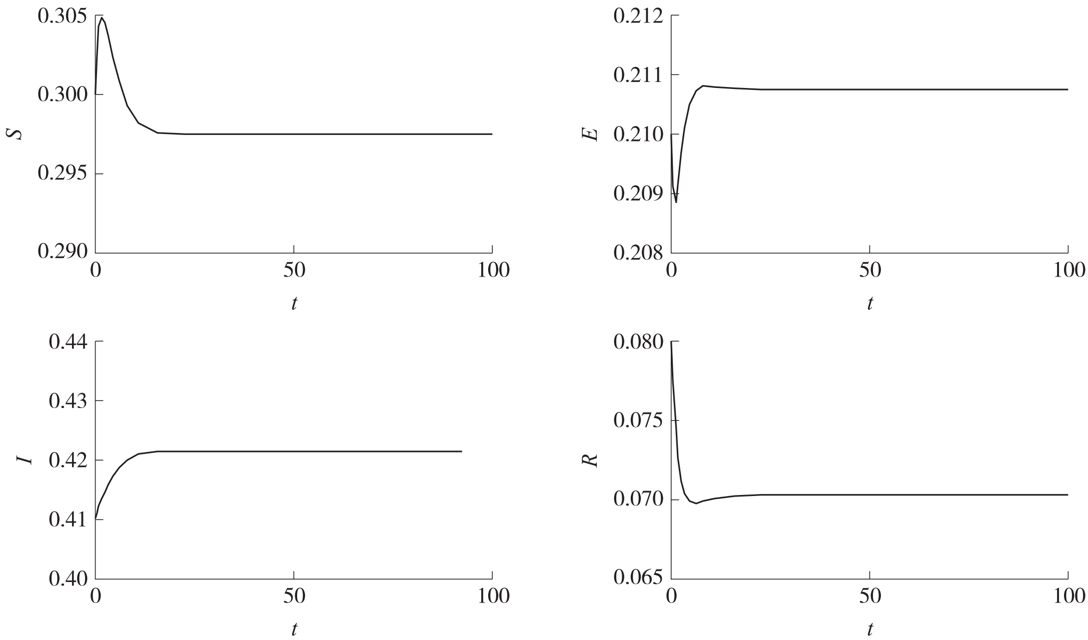

Fig. 1　The trajectories of S,E,I,R in system(1) with $\tau=0$ and $R_0=2.395$

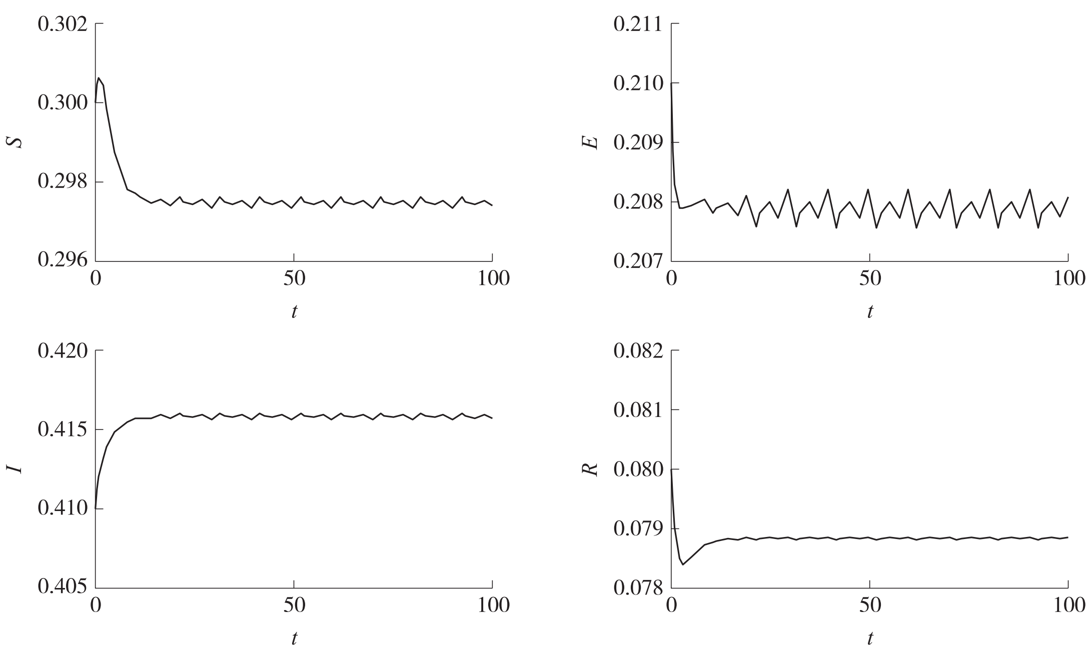

Fig. 2　The trajectories of S,E,I,R in system(1) with $\tau=1$ and $R_0=2.395$

4　Conclusions

In this paper, we consider an SEIRS epidemic model with vertical transmission, nonlinear incidence which has a more general form, and with a time delay in the removed class. Through mathematical analysis we can get that the dynamic behaviors of the model with time delay are different from those of the model without time delay. For the model without time delay, the disease free equilibrium is globally asymptotically stable when the basic reproduction number is smaller than one. When the basic reproduction number is bigger than one, there exists a unique endemic equilibrium which is locally asymptotically stable under a condition. For the model with time delay, the stability of the DFE depends on the time delay besides the basic reproduction number. In addition, regardless of the time delay length there exists a unique endemic equilibrium which is locally asymptotically stable under some conditions. From the mathematical analysis, it is easy to see the time delay can influence the dynamic behaviors of the SEIRS epidemic system. From (2) we know that R_0 is independent of the time delay. However, the stability of equilibria in the model (1) may depend on the time delay. It is obvious that the time delay can influence the stability of DFE and the endemic equilibrium according to Theorem 3 with Theorem 4.

To support this point, we perform some numerical simulations. We choose $h(S)=\frac{S}{1+\alpha S}$ and parameter values in the following:

$$\alpha=0.1;\beta=0.9;\varepsilon=0.6;p=0.3;\gamma=0.1;\eta=0.4;\mu=0.2.$$

Then $R_0=2.395$, $\tau_0=\frac{1}{\mu}\ln\frac{\eta}{\mu}=3.47$ and the condition (H) is satisfied in the two models. In Fig. 1, we let $\tau=0$ and we can see that the endemic equilibrium is stable, which supports the result of Theorem 2. However, in Fig. 2, we let $\tau=1<\tau_0$ and we can see oscillations can occur during $0<\tau<3.47$. This implies that the time delay can influence the dynamic behaviors of the SEIRS system (1).

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