# The Least Common Multiple of Arithmetic Progressions

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**Abstract**: For a positive arithmetic progression $U=\{u+d,u+2d,\cdots,u+nd\}$, $(u,d)=1$, we study the logarithm of the least common multiple of subsets of the set $U$. We show that for any $0<\theta<1$, $\log lcm\{a:a\in A\}=(1-\theta)n^{\theta}\log n+o(n^{\theta})$ for almost all sets $A\subset U$ of size $[n^{\theta}]$.

**Key words**: the least common multiple, arithmetic progression, probability

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# 算术级数子集的最小公倍数

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[摘要] 运用概率的方法研究了算术级数 $U$ 的子集的最小公倍数. 证明了对任意的 $0<\theta<1$,对几乎所有长度为 $[n^{\theta}]$ 的 $U$ 的子集 $A$,都有 $\log lcm\{a:a\in A\}=(1-\theta)n^{\theta}\log n+o(n^{\theta})$.

[关键词] 最小公倍数,算术级数,概率

For any set of positive integers $A$, let us denote

$$\psi(A)=\log lcm\{a:a\in A\}.$$

Let

$$U=\{u+d,u+2d,\cdots,u+nd\}$$

be an arithmetic progression, where $u$ and $d$ are positive integers with $(u,d)=1$. We also denote that

$$\bar{\psi}^{s}(n,k)=\frac{1}{\binom{n}{k}}\sum_{\substack{A\subset U\\ |A|=k}}\psi^{s}(A)$$

for $s=1,2$, which is the mean value of $\psi^{s}$ in the set of all subsets $A\subset U$ of size $k$. Recently Cilleruelo, Rué, Sarka and Zumalacarregui[1] proved that

$$\psi(A)=n\log 2+o(n)$$

for almost every set $A\subset\{1,\cdots,n\}$. They also studied the typical behavior of the logarithm of the least common multiple of sets of integers in $\{1,\cdots,n\}$ with prescribed size. For example, they proved that for any $0<\theta<1$,

$$\psi(A)=(1-\theta)n^{\theta}\log n+o(n^{\theta})$$

for almost all sets $A\subset\{1,\cdots,n\}$ of size $[n^{\theta}]$. For more results one may refer to [2-4].

In this paper we generalize the results in [1]. Specifically, we study the logarithm of the least common multiple of subsets of the positive arithmetic progression $U$ and obtain the following results:

**Theorem 1** Let $c>0$, $0<\theta<1$ and $k=cn^{\theta}+O(1)$. We have

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$$\bar{\psi}(n,k)=c(1-\theta)n^{\theta}\log n+O(n^{\theta})$$

for $0<\theta<1$, when $n\to\infty$. Furthermore, for any $\varepsilon>0$ we have that

$$\frac{|\{A:A\in U,|A|=k,|\psi(A)-\bar{\psi}(n,k)|<\varepsilon\bar{\psi}(n,k)\}|}{\binom{n}{k}}\to 1.$$

Theorem 1 implies that for $0<\theta<1$

$$\psi(A)\sim c(1-\theta)n^{\theta}\log n$$

for almost every set $A\subset U$ of size $[cn^{\theta}]$.

We shall apply the probabilistic approach to derive the main results. For a given $\delta$ with $0<\delta<1$, we select the elements of $A$ such that all its events $\{a\in A\}$ are independent and $P(a\in A)=\delta$ for every $a\in U$. Accordingly we denote this probability space by $S(n;\delta)$ which is the set of subsets of $U$ with the probability measure given by $P(X)=\delta^{|X|}(1-\delta)^{n-|X|}$ for any $X\subset U$.

Throughout this paper, we use the Vinogradov symbols $\gg$, $\ll$ and the Landau sym-bols $O$, $o$ with their regular meanings. We also use $[\ ]$ for the integer part function and $|\ |$ for the cardinality of one set.

Let us begin the proof of our main results. To prove Theorem 1 we need to prove the following two theorems firstly.

**Theorem 2** Let $c>0$ and $0<\theta<1$. In the probability space $S(n;cn^{\theta-1})$, the expected value of $\psi(A)$ with $A\subset U$ satisfies

$$E(\psi(A))=c(1-\theta)n^{\theta}\log n+O(n^{\theta})$$

for $0<\theta<1$, when $n\to\infty$.

**Theorem 3** In $S(n;\delta)$ the variance of $\psi(A)$ satisfies $V(\psi(A))\ll\delta n\log^2 n$.

## 1 Proof of Theorem 2

The following lemma is due to Cilleruelo, Rué, Sarka and Zumalacarregui[1].

**Lemma 1** [1, Lemma 2.1] For any set of positive integers $A$ we have

$$\psi(A)=\sum_{m}\Lambda(m)I_A(m),$$

where $\Lambda$ denotes the classical von Mangoldt function and

$$I_A(m)=\begin{cases}1, & \text{if } A\cap\{m,2m,3m,\cdots\}\neq\varnothing,\\ 0, & \text{otherwise.}\end{cases}$$

**Proof of Theorem 2** By Lemma 1, we have

$$E(\psi(A))=\sum_{1\leqslant m\leqslant u+nd}\Lambda(m)E(I_A(m)).$$

We define that

$$f(m)=|U\cap\{m,2m,\cdots\}|. \tag{1}$$

It is obvious that $f(m)=[n/m]$ or $f(m)=[n/m]+1$. On the other hand, we observe

$$E(I_A(m))=P(A\cap\{m,2m,\cdots\}\neq\varnothing)=1-\prod_{r\leqslant(u+nd)/m}P(rm\notin A)=\begin{cases}0 & \text{if } (d,m)>1,\\ 1-(1-\delta)^{f(m)}, & \text{if } (d,m)=1.\end{cases} \tag{2}$$

Thus

$$E(\psi(A))=\sum_{\substack{1\leqslant m\leqslant u+nd\\ (d,m)=1}}\Lambda(m)E(I_A(m))\leqslant\sum_{\substack{1\leqslant m\leqslant u+nd\\ (d,m)=1}}\Lambda(m)(1-(1-\delta)^{[n/m]+1})\leqslant$$

$$\sum_{1\leqslant m\leqslant u+nd}\Lambda(m)(1-(1-\delta)^{[n/m]+1})\leqslant\sum_{1\leqslant m\leqslant n}\Lambda(m)(1-(1-\delta)^{[n/m]+1})+\delta\sum_{n<m\leqslant u+nd}\Lambda(m)\leqslant$$

$$\sum_{1\leqslant m\leqslant n}\Lambda(m)(1-(1-\delta)^{[n/m]+1})+\delta(\psi(u+nd)-\psi(n)). \tag{3}$$

We split the left sum in (3) into intervals $J_r=\left(\frac{n}{r+1},\frac{n}{r}\right]$ since $[n/m]=r$ whenever $\frac{n}{r+1}<m\leqslant\frac{n}{r}$. Then

we have

$$\sum_{1\leqslant m\leqslant n}\Lambda(m)(1-(1-\delta)^{[n/m]+1})=\sum_{r\geqslant 1}(1-(1-\delta)^{r+1})\sum_{m\in J_r}\Lambda(m)=\sum_{r\geqslant 1}(1-(1-\delta)^{r+1})\left(\psi(n/r)-\psi\left(\frac{n}{r+1}\right)\right)=$$

$$\delta\sum_{r\geqslant 1}(1-\delta)^r\psi(n/r)+\delta\psi(n)=\delta n\sum_{r\geqslant 1}\frac{(1-\delta)^r}{r}+\delta n\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}+\delta\psi(n)=$$

$$\delta n\log\delta^{-1}+\delta n\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}+\delta\psi(n),\tag{4}$$

where $\varepsilon(x)=\frac{\psi(x)}{x}-1$, denotes the error term in the Prime Number Theory.

From (3) and (4) we get

$$E(\psi(A))\leqslant\delta n\log\delta^{-1}+\delta n\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}+\delta\psi(u+nd)\leqslant\delta n\log\delta^{-1}(1+\frac{1}{\log\delta^{-1}}\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}+$$
$$\frac{1}{\log\delta^{-1}}\frac{u+nd}{n}(1+\varepsilon(u+nd)))\leqslant\frac{\delta\log\delta^{-1}}{1-\delta}n(1+\frac{1}{\log\delta^{-1}}\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}+\frac{1}{\log\delta^{-1}}\frac{u+nd}{n}(1+\varepsilon(u+nd))).\tag{5}$$

Next we will estimate the lower bound of $E(\psi(A))$. With a similar discussion, we can obtain that

$$E(\psi(A))=\sum_{1\leqslant m\leqslant u+nd}\Lambda(m)E(I_A(m))\geqslant\sum_{\substack{1\leqslant m\leqslant n\\(d,m)=1}}\Lambda(m)(1-(1-\delta)^{[n/m]})\geqslant$$

$$\sum_{1\leqslant m\leqslant n}\Lambda(m)(1-(1-\delta)^{[n/m]})\sum_{\substack{t|d\\t|m}}\mu(t)\geqslant\sum_{t|d}\mu(t)\sum_{\substack{1\leqslant m\leqslant n\\t|m}}\Lambda(m)(1-(1-\delta)^{[n/m]})\geqslant$$

$$\sum_{1\leqslant m\leqslant n}\Lambda(m)(1-(1-\delta)^{[n/m]})-\sum_{p|d}\sum_{1\leqslant p^\alpha\leqslant n}\log p\geqslant\frac{\delta\log\delta^{-1}}{1-\delta}n\left(1+\frac{1}{\log\delta^{-1}}\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}\right)-\omega(d)\log n\geqslant$$

$$\frac{\delta\log\delta^{-1}}{1-\delta}n(1+\frac{1}{\log\delta^{-1}}\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}-\frac{1}{n\delta}\frac{1-\delta}{\log\delta^{-1}}w(d)\log n),\tag{6}$$

where $\omega(d)$ denotes the number of distinct prime factors of $d$.

By taking into account the error term in the Prime Number Theorem we can deduce that

$$\frac{1}{\log\delta^{-1}}\sum_{r\geqslant 1}\varepsilon(n/r)\frac{(1-\delta)^r}{r}\ll e^{-C_1\sqrt{\log(\delta n)}}\tag{7}$$

for some $C_1>0$. For more details of the proof of (7), one may refer to [1]. Thus we combine the above estimates (5), (6) and (7) to derive

$$E(\psi(A))=\frac{\delta\log\delta^{-1}}{1-\delta}n(1+O(1/\log n)).$$

Since $\delta=cn^{\theta-1}$, then we have

$$\frac{\log\delta^{-1}}{1-\delta}=((1-\theta)\log n-\log c)(1+O(n^{\theta-1}))$$

for $0<\theta<1$, which completes the proof of Theorem 2.

## 2 Proof of Theorem 3

Applying the similar methods of the proof of Proposition 3.1 in [1], we can deduce the following lemma which is omitted the proof in this section to avoid repeats.

**Lemma 2** For $s=1,2$, we have

$$\bar{\psi}^s(n,k)=E(\psi^s(A))+O(k^{s-1/2}\log^{s+1/2}n),$$

where $E(\psi(A))$ is the expectation of $\psi(A)$ in $S(n;k/n)$.

**Proof of Theorem 3** By the linearity of expectation we have that

$$V(\psi(A))=E(\psi^2(A))-(E(\psi(A)))^2=\sum_{\substack{1\leqslant m,l\leqslant u+nd\\(d,m)=1,(d,l)=1}}\Lambda(m)\Lambda(l)(E(I_A(m)I_A(l))-$$

$$E(I_A(m))E(I_A(l)))\leqslant\sum_{1\leqslant m,l\leqslant u+nd}\Lambda(m)\Lambda(l)(E(I_A(m)I_A(l))-E(I_A(m))E(I_A(l))).$$

Note that if $\Lambda(m)\Lambda(l)\neq 0$, then $l|m$, $m|l$ or $(m,l)=1$. Next we will calculate $E(I_A(m)I_A(l))$ in these cases.

If $l|m$ then

$$E(I_A(m)I_A(l))=1-(1-\delta)^{f(m)}.$$

If $(m,l)=1$ then

$$E(I_A(m)I_A(l))=1-(1-\delta)^{f(m)}-(1-\delta)^{f(l)}+(1-\delta)^{f(m)+f(l)-f(ml)}.$$

We observe that both of the above two relations are subsumed in

$$E(I_A(m)I_A(l))=1-(1-\delta)^{f(m)}-(1-\delta)^{f(l)}+(1-\delta)^{f(m)+f(l)-f([m,l])}.$$

On the other hand by (1) we have

$$E(I_A(m))E(I_A(l))=(1-(1-\delta)^{f(m)})(1-(1-\delta)^{f(l)}).$$

Without loss of generality we may assume that $l\leqslant m$ and use the inequality $1-(1-x)^r\leqslant rx$ to obtain

$$\Lambda(m)\Lambda(l)(E(I_A(m)I_A(l))-E(I_A(m))E(I_A(l)))=\Lambda(m)\Lambda(l)(1-(1-\delta)^{f([m,l])})(1-\delta)^{f(m)+f(l)-f([m,l])}\leqslant$$
$$\Lambda(m)\Lambda(l)\delta f([m,l])\ll\delta n(m,l)\frac{\Lambda(m)}{m}\frac{\Lambda(l)}{l}.$$

Thus, we have

$$V(\psi(A))\ll\delta n\sum_{1\leqslant l\leqslant m\leqslant u+nd}(m,l)\frac{\Lambda(m)}{m}\frac{\Lambda(l)}{l}\ll\delta n\sum_{1<p\leqslant u+nd}\sum_{1\leqslant j\leqslant k}\frac{\log p}{p^j}\frac{\log p}{p^k}p^j+\delta n\left(\sum_{1<p\leqslant u+nd}\sum_{k\geqslant 1}\frac{\log p}{p^k}\right)^2\ll$$
$$\delta n\sum_{1<p\leqslant u+nd}\sum_{k\geqslant 1}\frac{k\log^2 p}{p^k}+\delta n\left(\sum_{1<p\leqslant u+nd}\log p\sum_{k\geqslant 1}\frac{1}{p^k}\right)^2\ll\delta n\sum_{1<p\leqslant u+nd}\frac{\log^2 p}{p}\sum_{k\geqslant 1}\frac{k}{2^{k-1}}+\delta n\left(\sum_{1<p\leqslant u+nd}\frac{\log p}{p}\sum_{k\geqslant 1}\frac{1}{2^{k-1}}\right)^2\ll$$
$$\delta n\log^2(u+nd) \tag{8}$$

which completes the proof of Theorem 3.

## 3 Proof of Theorem 1

Now we will use Theorem 2, Theorem 3 and Lemma 2 to prove Theorem 1.

For $k\sim cn^\theta$ and $\delta=cn^{\theta-1}$ we have

$$\binom{n}{k}^{-1}\sum_{A\in U,|A|=k}(\psi(A)-\bar{\psi}(n,k))^2=\overline{\psi^2}(n,k)-\bar{\psi}^2(n,k)=V(\psi(A))+(\overline{\psi^2}(n,k)-E(\psi^2(A)))+$$
$$(E(\psi(A))-\bar{\psi}(n,k))(E(\psi(A))+\bar{\psi}(n,k))\ll n^\theta\log^2 n+n^{3\theta/2}\log^{5/2}n+(n^{\theta/2}\log^{3/2}n)(n^\theta\log n)\ll$$
$$n^{3\theta/2}\log^{5/2}n.$$

Thus from Chebyshev's inequality we get that

$$\frac{|\{A:A\in U,|A|=k,|\psi(A)-\bar{\psi}(n,k)|\geqslant\varepsilon\bar{\psi}(n,k)\}|}{\binom{n}{k}}\ll\frac{n^{3\theta/2}\log^{5/2}n}{(\varepsilon\bar{\psi}(n,k))^2}\ll\frac{n^{3\theta/2}\log^{5/2}n}{(\varepsilon n^\theta\log n)^2}\ll\frac{\log^{1/2}n}{\varepsilon^2 n^{\theta/2}}\to 0,$$

which completes the proof of Theorem 1.

**Remark** The proofs of our theorems can be modified to deduce the similar results for any positive sequences.

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