A New Preconditioned AOR Iterative Method for Linear System with M-Matrices

Tan Xueyuan

(Jiangsu Key Laboratory for NSLSCS,School of Mathematical Sciences,Nanjing Normal University,Nanjing 210023,China)

Abstract:The purpose of this paper is to investigate the preconditioned AOR method with a new preconditioner denoted as $I+S_\alpha+S_M+S_\delta$ for M-matrix. The new preconditioner is constructed by considering the largest absolute value of the upper triangular part,the secondary diagonal and the last column of the coefficient matrix A. We prove that the rate of the AOR iterative method can be accelerated,and give the comparison with other three preconditioners to show the new preconditioner is more effective. Numerical example demonstrates the effectiveness of this preconditioning scheme.

Key words:linear system,AOR iterative method,preconditioner,M-matrix

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关于 M 矩阵的线性系统一个新的 AOR 预处理方法

谈雪媛

(南京师范大学数学科学学院,江苏省大规模系统数值模拟重点实验室,南京 210023)

[**摘要**] 本文研究 M 矩阵的预处理 AOR 方法,并给出新的预处理子 $I+S_\alpha+S_M+S_\delta$. 新的预处理子是基于系数矩阵 A 的上三角部分绝对值最大元素,次对角元素以及最后一列元素构建. 我们证明此法将加速 AOR 迭代速率,并通过与其他三个预处理子的比较说明新的预处理子更有效. 数值例子验证了此预处理方法的有效性.

[**关键词**] 线性系统,AOR 迭代方法,预处理子,M-矩阵

Many preconditioned iterative methods were proposed to improve the efficiency and robustness of basic iterative methods,all of which transformed the original nonsingular linear system

$$Ax=b \tag{1}$$

into the preconditioned form

$$PAx=Pb,$$

where $A=(a_{ij})\in C^{n\times n}$, $b\in R^n$ are given,the preconditioner $P\in R^{n\times n}$ is chosen as a nonsingular and nonnegative matrix with unit diagonal entries,and $x\in C^n$ is unknown. For ease of presentation,we assume that A possesses unit diagonal entries and is split as $A=I-L-U$,where I, L and U are respectively diagonal,strictly lower and strictly upper triangular parts of A. The standard Accelerated Overrelaxation(AOR)iterative method[1] for solving (1) is denoted as

$$x^{k+1}=L_{\gamma,\omega}x^k+(I-\gamma L)^{-1}\omega b,\quad k=0,1,2,\cdots \tag{2}$$

with the iteration matrix

$$L_{\gamma,\omega}=(I-\gamma L)^{-1}[(1-\omega)I+(\omega-\gamma)L+\omega U]. \tag{3}$$

As is known to all that for certain values of ω (relaxation parameter) and γ (acceleration parameter) in

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Corresponding author:Tan Xueyuan,lecturer,majored in numerical linear algebra. E-mail:tanxueyuan@njnu.edu.cn

(2), we can get some well-known basic iterative methods. For instance, we obtain the Jacobi iterative method if $\omega=1$, $\gamma=0$, the JOR iterative method if $\gamma=0$, the Gauss-Seidel(GS) iterative method if $\omega=\gamma=1$ and the Successive Overrelaxation(SOR)[2] iterative method if $\omega=\gamma$.

The selection strategies of preconditioners of the preconditioned iterative methods were discussed in the past and recent years. We list some of them which will be mentioned in this paper. In 1991, Gunawardena et al.[3] suggested the modified Gauss-Seidel method with the preconditioner $\boldsymbol{I}+\boldsymbol{S}$, Kohno et al.[4] then gave the generalized preconditioner $\boldsymbol{I}+\boldsymbol{S}_\alpha$ in 1997, with their preconditioner Wu et al.[5] presented the preconditioned AOR iterative method and corresponding convergence results in 2007. Another advantageous idea came from Kotakemori et al.[6] in 2002, they got the comparison theorem for the modified Gauss-Seidel method by proposing the preconditioner $\boldsymbol{I}+\boldsymbol{S}_m$ where $\boldsymbol{S}_m$ is constructed by only the largest element at each row of the upper triangular part of $\boldsymbol{A}$, in 2004 Morimoto et al.[7] considered the preconditioner $\boldsymbol{I}+\boldsymbol{S}+\boldsymbol{S}_m$ to ameliorate the results in [6] and in 2009 Zheng et al. suggested two new preconditioners observing on the last row to deal with the drawback in [6].

In this work, motivated by the above tactics but different from them, we construct a new preconditioner by considering the largest absolute value of the upper triangular part(not just the largest element at each row of the upper triangular part of $\boldsymbol{A}$ in [6]), the secondary diagonal and the last column of the coefficient matrix $\boldsymbol{A}$, which is denoted as $\overset{\cup}{\boldsymbol{P}}=\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta$. We prove that the rate of the AOR iterative method can be accelerated, and give the comparisons with other three preconditioners given before to show our preconditioner is more effective.

For convenience, we introduce some notations, some known concepts and results below. Throughout this paper, we use symbol $\rho(\boldsymbol{A})$ to denote the spectral radius of $\boldsymbol{A}$, we write $\boldsymbol{A}\geqslant 0$, $\boldsymbol{A}>0$, $\boldsymbol{A}\gg 0$ if all elements of $\boldsymbol{A}$ are nonnegative, nonnegative but at least a positive element and positive, respectively. We can also identify the n-dimensional vector with $n\times 1$ matrix in order to define $\boldsymbol{x}\geqslant 0$, $\boldsymbol{x}>0$, $\boldsymbol{x}\gg 0$ individually.

Definition 1 Let $\boldsymbol{A}=(a_{ij})\in\boldsymbol{R}^{n\times n}$. Then $\boldsymbol{A}$ is called

(1) a nonnegative matrix if $a_{ij}\geqslant 0$;

(2) an Z-matrix if $a_{ij}\leqslant 0$ for all $i\neq j$;

(3) an L-matrix if $\boldsymbol{A}$ is an Z-matrix and $a_{ii}>0$;

(4) a nonsingular M-matrix if $\boldsymbol{A}$ is a nonsingular L-matrix and $\boldsymbol{A}^{-1}\geqslant 0$.

Definition 2 Let $\boldsymbol{M},\boldsymbol{N}\in\boldsymbol{R}^{n\times n}$. Then $\boldsymbol{A}=\boldsymbol{M}-\boldsymbol{N}$ is called

(1) a regular splitting if $\boldsymbol{M}^{-1}\geqslant 0$ and $\boldsymbol{N}\geqslant 0$;

(2) an M-splitting if $\boldsymbol{M}$ is a nonsingular M-matrix and $\boldsymbol{N}\geqslant 0$;

(3) a convergent splitting if $\boldsymbol{M}$ is nonsingular and $\rho(\boldsymbol{M}^{-1}\boldsymbol{N})<1$.

Lemma 1[8] Let $\boldsymbol{A}\in\boldsymbol{R}^{n\times n}$ be an Z-matrix. Then next three statements are equivalent:

(1) $\boldsymbol{A}$ is a nonsingular M-matrix;

(2) There is a vector $\boldsymbol{x}\gg 0$ such that $\boldsymbol{A}\boldsymbol{x}\gg 0$;

(3) Any regular splitting is convergent.

Lemma 2 Let $\boldsymbol{A}\in\boldsymbol{R}^{n\times n}$ be a nonnegative matrix. Then:

(1)[8] If $\alpha\boldsymbol{x}\leqslant\boldsymbol{A}\boldsymbol{x}$ for some $\boldsymbol{x}>0$, then $\alpha\leqslant\rho(\boldsymbol{A})$;

(2)[9] $\rho(\boldsymbol{A})$ is the eigenvalue of $\boldsymbol{A}$ and there exists an eigenvector $\boldsymbol{x}>0$ corresponds to $\rho(\boldsymbol{A})$.

1 AOR Iterative Method with the Preconditioner $\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta$

Let us consider the preconditioned linear system

$$\overset{\cup}{\boldsymbol{P}}\boldsymbol{A}\boldsymbol{x}=\overset{\cup}{\boldsymbol{P}}\boldsymbol{b},$$

where $\boldsymbol{A}=(a_{ij})\in\boldsymbol{R}^{n\times n}$ is nonsingular with unit diagonal entries, $\boldsymbol{b}\in\boldsymbol{R}^n$,

$$\overset{\cup}{\boldsymbol{P}}=(p_{ij})=\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta$$

with

$$\boldsymbol{S}_\alpha=(s_{ij})=\begin{cases}-\alpha_i a_{ii+1}, & 1\leqslant i\leqslant n-1, j=i+1, 0\leqslant\alpha_i\leqslant 1,\\ 0, & \text{otherwise},\end{cases}$$

$$\boldsymbol{S}_M=(\overset{\cup}{s}_{ij})=\begin{cases}a, & j=j_i,\\ 0, & \text{otherwise},\end{cases}$$

where $j_i=\min I_i$, $I_i=\{j:|a_{ij}|$ is maximal for $i+2\leqslant j\leqslant n\}$, $1\leqslant i\leqslant n-1$, a are defined as

$$a=\begin{cases}M, & \text{if } \alpha_i a_{ii+1}a_{i+1j_i}-a_{ij_i}\geqslant M,\\ -a_{ij_i}, & \text{otherwise},\end{cases}$$

with

$$M=\max\{|a_{ij}|, i\neq n, j>i\},\quad \boldsymbol{S}_\delta=(\hat{s}_{ij})=\begin{cases}b, & j=n \text{ and } j_i\neq n,\\ 0, & \text{otherwise},\end{cases}$$

where

$$b=\begin{cases}M, & \text{if } \alpha_i a_{ii+1}a_{i+1n_i}-aa_{j_in}-a_{in}\geqslant M,\\ -a_{in}, & \text{otherwise}.\end{cases}$$

We split $\overset{\cup}{\boldsymbol{A}}=(\overset{\cup}{a}_{ij})=\overset{\cup}{\boldsymbol{P}}\boldsymbol{A}$ as $\overset{\cup}{\boldsymbol{A}}=\overset{\cup}{\boldsymbol{D}}-\overset{\cup}{\boldsymbol{L}}-\overset{\cup}{\boldsymbol{U}}$, where $\overset{\cup}{\boldsymbol{D}}, \overset{\cup}{\boldsymbol{L}}, \overset{\cup}{\boldsymbol{U}}$ are respectively diagonal, strictly lower and strictly upper triangular matrices. Then the preconditioned AOR iterative method is expressed as

$$x^{k+1}=\boldsymbol{J}_{\gamma,\omega}x^k+\omega(\overset{\cup}{\boldsymbol{D}}-\gamma\overset{\cup}{\boldsymbol{L}})^{-1}\overset{\cup}{\boldsymbol{P}}\boldsymbol{b},\quad k=0,1,2,\cdots,$$

where

$$\boldsymbol{J}_{\gamma,\omega}=(\overset{\cup}{\boldsymbol{D}}-\gamma\overset{\cup}{\boldsymbol{L}})^{-1}[(1-\omega)\overset{\cup}{\boldsymbol{D}}+(\omega-\gamma)\overset{\cup}{\boldsymbol{L}}+\omega\overset{\cup}{\boldsymbol{U}}]. \tag{4}$$

By computing, $\overset{\cup}{a}_{ij}$ then can be expressed as

$$\overset{\cup}{a}_{ij}=\begin{cases}1+\sum\limits_{k\neq i}^{n}p_{ik}a_{ki}, & 1\leqslant i=j\leqslant n,\\ p_{ij}+\sum\limits_{k\neq j}^{n}p_{ik}a_{kj}, & 1\leqslant i\neq j\leqslant n.\end{cases}$$

For simplicity, we give the unified denotation $\boldsymbol{P}^*=(p_{ij}^*)$ as the preconditioned matrix for $\boldsymbol{I}+\boldsymbol{S}$[3], $\boldsymbol{I}+\boldsymbol{S}_\alpha$[4], $\boldsymbol{I}+\boldsymbol{S}_m$[6] and $\boldsymbol{I}+\boldsymbol{S}+\boldsymbol{S}_m$[7]. Here

$$\boldsymbol{S}_m=(\overset{\vee}{s}_{ij})=\begin{cases}-a_{ik_i}, & j>i,\\ 0, & \text{otherwise},\end{cases}$$

where $k_i=\min U_i$, $U_i=\{j:|a_{ij}|$ is maximal for $i+1\leqslant j\leqslant n\}$, $1\leqslant i\leqslant n-1$. Since $\boldsymbol{S}_\alpha$ becomes $\boldsymbol{S}$ if $\alpha_i=1, i=1,2,\cdots,n$, we only need to consider $\boldsymbol{S}_\alpha$ when $\boldsymbol{S}$ appears.

Similarly, we split $\boldsymbol{A}^*=(a_{ij}^*)=\boldsymbol{P}^*\boldsymbol{A}$ as $\boldsymbol{A}^*=\boldsymbol{D}^*-\boldsymbol{L}^*-\boldsymbol{U}^*$, where $\boldsymbol{D}^*, \boldsymbol{L}^*, \boldsymbol{U}^*$ are respectively diagonal, strictly lower and strictly upper triangular matrices, then the preconditioned AOR iterative method is

$$\boldsymbol{x}^{k+1}=\boldsymbol{J}_{\gamma,\omega}^*\boldsymbol{x}^k+\omega(\boldsymbol{D}^*-\gamma\boldsymbol{L}^*)^{-1}\boldsymbol{P}^*\boldsymbol{b},\quad k=0,1,2,\cdots,$$

with the iteration matrix

$$\boldsymbol{J}_{\gamma,\omega}^*=(\boldsymbol{D}^*-\gamma\boldsymbol{L}^*)^{-1}[(1-\omega)\boldsymbol{D}^*+(\omega-\gamma)\boldsymbol{L}^*+\omega\boldsymbol{U}^*].$$

We first show a fundamental lemma utilized in this paper.

Lemma 3[10] Let $\boldsymbol{A}\in\boldsymbol{R}^{n\times n}$ be a nonsingular M-matrix. If $\boldsymbol{P}=(\hat{p}_{ij})\in\boldsymbol{R}^{n\times n}$ is a nonsingular nonnegative preconditioner matrix such that $\hat{p}_{ii}=1$, $i=1,2,\cdots,n$, and $\hat{p}_{ij}+\sum\limits_{k=1,k\neq j}^{n}\hat{p}_{ij}a_{kj}\leqslant 0$, $1\leqslant i\neq j\leqslant n$, then the following statements are true:

(1) $\boldsymbol{PA}$ is also a nonsingular M-matrix.

(2) If $\rho(\overset{\cup}{\boldsymbol{L}}_{\gamma,\omega})<1$, then $\rho(\overset{\cup}{\boldsymbol{L}}_{\gamma,\omega})\leqslant\rho(\boldsymbol{L}_{\gamma,\omega})<1$, where $\overset{\cup}{\boldsymbol{L}}_{\gamma,\omega}$ is the preconditioned AOR iteration matrix and

$\boldsymbol{L}_{\gamma,\omega}$ is defined in (3).

Remark 1 Noting Definition 1, we can give the equivalent expression of the first part of this lemma: for a nonsingular M-matrix $\boldsymbol{A}$, $\boldsymbol{PA}$ is also a nonsingular M-matrix if $\boldsymbol{PA}$ is an Z-matrix.

Theorem 1 If $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$ is a nonsingular M-matrix, then $\breve{\boldsymbol{P}}\boldsymbol{A} = (\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta)\boldsymbol{A}$ is a nonsingular M-matrix.

Proof By the result of Lemma 3, we only need to prove that $\breve{\boldsymbol{P}}\boldsymbol{A}$ is an Z-matrix, i. e., to prove that all the off-diagonal entries of $\breve{\boldsymbol{P}}\boldsymbol{A}$ written as $\breve{a}_{ij}(i\neq j)$ are nonpositive.

If $p_{ij}=0$, obviously $\breve{a}_{ij} = \sum\limits_{k\neq j} p_{ik}a_{kj} \leqslant 0$. If $p_{ij}\neq 0$, then there are four possibilities:

(1) when $j=i+1$,

$$\breve{a}_{ij} = p_{ii+1}+p_{ii}a_{ii+1}+p_{ii+2}a_{i+2,i+1}+\cdots+p_{in}a_{ni+1} = -\alpha_i a_{ii+1}+a_{ii+1}+p_{ii+2}a_{i+2,i+1}+\cdots+p_{in}a_{ni+1} = (1-\alpha_i)a_{ii+1}+p_{ii+2}a_{i+2,i+1}+\cdots+p_{in}a_{ni+1} \leqslant 0.$$

(2) when $j_i<n$ and $j\neq n$,

$$\breve{a}_{ij} = a+p_{ii}a_{ij_i}+p_{ii+1}a_{i+1j_i}+\cdots+p_{ij_i-1}a_{j_i-1j_i}+p_{ij_i+1}a_{j_i+1j_i}+\cdots+p_{in}a_{nj_i} = a+a_{ij_i}-\alpha_i a_{ii+1}a_{i+1j_i}+ba_{nj_i} \leqslant 0.$$

(3) when $j_i<n$ and $j=n$,

$$\breve{a}_{in} = p_{in}+p_{ii}a_{in}+p_{ii+1}a_{i+1n}+p_{ij_i}a_{j_in} = b+a_{in_i}-\alpha_i a_{ii+1}a_{i+1n}+aa_{j_in} \leqslant 0.$$

(4) when $j_i=n$ and $j=n$,

$$\breve{a}_{in} = p_{in}+p_{ii}a_{in}+p_{ii+1}a_{i+1n} = a+a_{in_i}-\alpha_i a_{ii+1}a_{i+1n} \leqslant 0.$$

The last inequalities of (2), (3) and (4) hold because of the selection of a.

Thus, we have the statement of Theorem 1.

Remark 2 It can be verified similarly that $\boldsymbol{P}^*\boldsymbol{A}$ is a nonsingular M-matrix if $\boldsymbol{A}$ is a nonsingular M-matrix.

Theorem 2 If $\boldsymbol{A}$ is an Z-matrix and with $\rho(\boldsymbol{L}_{\gamma,\omega})<1$ and $0\leqslant\gamma\leqslant\omega\leqslant 1$, $\omega\neq 0$, then the new preconditioned AOR method satisfies $\rho(\boldsymbol{J}_{\gamma,\omega})\leqslant\rho(\boldsymbol{L}_{\gamma,\omega})<1$, where $\boldsymbol{L}_{\gamma,\omega}$ and $\boldsymbol{J}_{\gamma,\omega}$ are defined in (3) and (4).

Proof The result follows directly from Theorem 1 and Lemma 3.

Remark 3 The result and proof can apply to $\rho(\boldsymbol{J}^*_{\gamma,\omega})$ since $\boldsymbol{P}^*\boldsymbol{A}$ is an M-matrix by Remark 2.

Let $\breve{\boldsymbol{A}} = \breve{\boldsymbol{P}}\boldsymbol{A} = \breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$, $\boldsymbol{A}^* = \boldsymbol{P}^*\boldsymbol{A} = \boldsymbol{M}^*-\boldsymbol{N}^*$, where

$$\breve{\boldsymbol{M}} = \frac{1}{\omega}(\breve{\boldsymbol{D}}-\gamma\breve{\boldsymbol{L}}),\ \boldsymbol{M}^* = \frac{1}{\omega}(\boldsymbol{M}^*-\gamma\boldsymbol{L}^*),$$

and

$$\breve{\boldsymbol{N}} = \frac{1}{\omega}[(1-\omega)\breve{\boldsymbol{D}}+(\omega-\gamma)\breve{\boldsymbol{L}}+\omega\breve{\boldsymbol{U}}],\ \boldsymbol{N}^* = \frac{1}{\omega}[(1-\omega)\boldsymbol{D}^*+(\omega-\gamma)\boldsymbol{L}^*+\omega\boldsymbol{U}^*].$$

Then we obtain the following results.

Theorem 3 If $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$ is a nonsingular M-matrix and $0\leqslant\gamma\leqslant\omega\leqslant 1$, $\omega\neq 0$, then

(1) $\breve{\boldsymbol{A}} = \breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$ is an M-splitting;

(2) $(\breve{\boldsymbol{M}})^{-1} \geqslant (\boldsymbol{M}^*)^{-1}$.

Proof Let $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3$ be diagonal matrices of $\boldsymbol{S}_\alpha\boldsymbol{L}, \boldsymbol{S}_M\boldsymbol{L}, \boldsymbol{S}_\delta\boldsymbol{L}$ and $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3$ be strictly lower triangular matrices of $\boldsymbol{S}_\alpha\boldsymbol{L}, \boldsymbol{S}_M\boldsymbol{L}, \boldsymbol{S}_\delta\boldsymbol{L}$ respectively. Then

$$\breve{\boldsymbol{P}}\boldsymbol{A} = \breve{\boldsymbol{D}}-\breve{\boldsymbol{L}}-\breve{\boldsymbol{U}} = (\boldsymbol{I}-\boldsymbol{E}_1-\boldsymbol{E}_2-\boldsymbol{E}_3)-(\boldsymbol{L}+\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3)-(\boldsymbol{U}-\boldsymbol{S}_\alpha-\boldsymbol{S}_M-\boldsymbol{S}_\delta+\boldsymbol{S}_\alpha\boldsymbol{U}+\boldsymbol{S}_M\boldsymbol{U}+\boldsymbol{S}_\delta\boldsymbol{U}),$$

$$(\breve{\boldsymbol{M}})^{-1} = \omega[\boldsymbol{I}+\gamma(\boldsymbol{I}-\boldsymbol{E}_1-\boldsymbol{E}_2-\boldsymbol{E}_3)^{-1}(\boldsymbol{L}+\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3)+\gamma^2(\boldsymbol{I}-\boldsymbol{E}_1-\boldsymbol{E}_2-\boldsymbol{E}_3)^{-2}+\cdots+(\boldsymbol{L}+\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3)^2+ \gamma^{n-1}(\boldsymbol{I}-\boldsymbol{E}_1-\boldsymbol{E}_2-\boldsymbol{E}_3)^{1-n}(\boldsymbol{L}+\boldsymbol{F}_1+\boldsymbol{F}_2+\boldsymbol{F}_3)^{n-1}](\boldsymbol{I}-\boldsymbol{E}_1-\boldsymbol{E}_2-\boldsymbol{E}_3)^{-1} \geqslant 0.$$

Obviously, $\breve{\boldsymbol{M}}$ is an L-matrix and $\breve{\boldsymbol{N}}\geqslant 0$. Hence $\breve{\boldsymbol{A}} = \breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$ is an M-splitting.

Next we discuss the case when $\boldsymbol{P}^*\boldsymbol{A} = \boldsymbol{D}^*-\boldsymbol{L}^*-\boldsymbol{U}^*$ from the three different expressions of $\boldsymbol{P}^*$.

(a) If we take

$$\boldsymbol{P}^{*}\boldsymbol{A}=(\boldsymbol{I}+\boldsymbol{S}_{\alpha})(\boldsymbol{I}-\boldsymbol{L}-\boldsymbol{U})=(\boldsymbol{I}-\boldsymbol{E}_{1})-(\boldsymbol{L}+\boldsymbol{F}_{1})-(\boldsymbol{U}-\boldsymbol{S}_{\alpha}+\boldsymbol{S}_{\alpha}\boldsymbol{U}),$$

$$(\boldsymbol{M}^{*})^{-1}=\omega[(\boldsymbol{I}-\boldsymbol{E}_{1})-\gamma(\boldsymbol{L}+\boldsymbol{F}_{1})]^{-1}=\omega[\boldsymbol{I}-\gamma(\boldsymbol{I}-\boldsymbol{E}_{1})^{-1}(\boldsymbol{L}+\boldsymbol{F}_{1})]^{-1}(\boldsymbol{I}-\boldsymbol{E}_{1})^{-1},$$

it follows that

$$(\breve{\boldsymbol{M}})^{-1}\geqslant\omega[\boldsymbol{I}+\gamma(\boldsymbol{I}-\boldsymbol{E}_{1})^{-1}(\boldsymbol{L}+\boldsymbol{F}_{1})+\gamma(\boldsymbol{I}-\boldsymbol{E}_{1})^{-2}(\boldsymbol{L}+\boldsymbol{F}_{1})^{2}+\cdots+\gamma^{n-1}(\boldsymbol{I}-\boldsymbol{E}_{1})^{1-n}(\boldsymbol{L}+\boldsymbol{F}_{1})^{n-1}](\boldsymbol{I}-\boldsymbol{E}_{1})^{-1}=(\boldsymbol{M}^{*})^{-1}.$$

(b) If we take $\boldsymbol{P}^{*}=\boldsymbol{I}+\boldsymbol{S}_{m}$ and denote $\boldsymbol{S}_{m}\boldsymbol{L}=\boldsymbol{E}_{2}^{*}+\boldsymbol{F}_{2}^{*}$ are respectively diagonal matrix and strictly lower triangular matrix, then

$$\boldsymbol{P}^{*}\boldsymbol{A}=(\boldsymbol{I}+\boldsymbol{S}_{m})(\boldsymbol{I}-\boldsymbol{L}-\boldsymbol{U})=(\boldsymbol{I}-\boldsymbol{E}_{2})-(\boldsymbol{L}+\boldsymbol{F}_{2}^{*})-(\boldsymbol{U}-\boldsymbol{S}_{m}+\boldsymbol{S}_{m}\boldsymbol{U}),$$

here

$$(\boldsymbol{M}^{*})^{-1}=\omega[(\boldsymbol{I}-\boldsymbol{E}_{2}^{*})-\gamma(\boldsymbol{L}+\boldsymbol{F}_{2}^{*})]^{-1}=\omega[\boldsymbol{I}-\gamma(\boldsymbol{I}-\boldsymbol{E}_{2}^{*})^{-1}(\boldsymbol{L}+\boldsymbol{F}_{2}^{*})]^{-1}(\boldsymbol{I}-\boldsymbol{E}_{2}^{*})^{-1},$$

since $\boldsymbol{E}_{2}^{*}\leqslant\boldsymbol{E}_{2}$, $\boldsymbol{F}_{2}^{*}\leqslant\boldsymbol{F}_{2}$, it holds that

$$(\breve{\boldsymbol{M}})^{-1}\geqslant\omega[\boldsymbol{I}+\gamma(\boldsymbol{I}-\boldsymbol{E}_{2}^{*})^{-1}(\boldsymbol{L}+\boldsymbol{F}_{2}^{*})+\gamma(\boldsymbol{I}-\boldsymbol{E}_{2}^{*})^{-2}(\boldsymbol{L}+\boldsymbol{F}_{2}^{*})^{2}+\cdots+\gamma^{n-1}(\boldsymbol{I}-\boldsymbol{E}_{2}^{*})^{1-n}(\boldsymbol{L}+\boldsymbol{F}_{1})^{n-1}](\boldsymbol{I}-\boldsymbol{E}_{2}^{*})^{-1}=(\boldsymbol{M}^{*})^{-1}.$$

(c) If we take $\boldsymbol{P}^{*}=\boldsymbol{I}+\boldsymbol{S}_{\alpha}+\boldsymbol{S}_{m}$, then

$$\boldsymbol{P}^{*}\boldsymbol{A}=(\boldsymbol{I}+\boldsymbol{S}_{\alpha}+\boldsymbol{S}_{m})(\boldsymbol{I}-\boldsymbol{L}-\boldsymbol{U})=(\boldsymbol{I}-\boldsymbol{E}_{1}-\boldsymbol{E}_{2}^{*})-(\boldsymbol{L}+\boldsymbol{F}_{1}+\boldsymbol{F}_{2}^{*})-(\boldsymbol{U}-\boldsymbol{S}_{\alpha}-\boldsymbol{S}_{m}+\boldsymbol{S}_{\alpha}\boldsymbol{U}+\boldsymbol{S}_{m}\boldsymbol{U}),$$

we obtain that

$$(\breve{\boldsymbol{M}})^{-1}\geqslant\omega[\boldsymbol{I}+\gamma(\boldsymbol{I}-\boldsymbol{E}_{1}-\boldsymbol{E}_{2}^{*})^{-1}(\boldsymbol{L}+\boldsymbol{F}_{1}+\boldsymbol{F}_{2}^{*})+\gamma(\boldsymbol{I}-\boldsymbol{E}_{1}-\boldsymbol{E}_{2}^{*})^{-2}(\boldsymbol{L}+\boldsymbol{F}_{1}+\boldsymbol{F}_{2}^{*})^{2}+\cdots+$$
$$\gamma^{n-1}(\boldsymbol{I}-\boldsymbol{E}_{1}-\boldsymbol{E}_{2}^{*})^{1-n}(\boldsymbol{L}+\boldsymbol{F}_{1}+\boldsymbol{L}_{2}^{*})^{n-1}](\boldsymbol{I}-\boldsymbol{E}_{1}-\boldsymbol{E}_{2}^{*})^{-1}=(\boldsymbol{M}^{*})^{-1}.$$

From (a), (b), (c), the second part of this theorem follows.

Remark 4 (1) If we take $\alpha_{i}=1$ for all i, then the theorem is also true for the preconditioners $\boldsymbol{I}+\boldsymbol{S}$ and $\boldsymbol{I}+\boldsymbol{S}+\boldsymbol{S}_{m}$ presented in [3] and [7].

(2) $\breve{\boldsymbol{A}}=\breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$ is an M-splitting implies that $(\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}}\geqslant 0$. Thus, by Lemma 2, $\rho((\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}})$ is the eigenvalue of $(\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}}$ and there exists an eigenvector $\boldsymbol{x}>0$ according to $\rho((\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}})$.

(3) It is easy to see that $\breve{\boldsymbol{A}}=\breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$ is a regular splitting, then by Lemma 1, $\breve{\boldsymbol{A}}=\breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$ is a convergent splitting.

In the following theorem we supply the comparison theorem to show that our preconditioner is superior in increasing the convergence rate of AOR iterative method.

Theorem 4 Let $\boldsymbol{A}\in\boldsymbol{R}^{n\times n}$ be a nonsingular M-matrix and $\rho(\boldsymbol{L}_{\gamma,\omega})<1$, suppose that $\breve{\boldsymbol{A}}\boldsymbol{x}\geqslant\boldsymbol{A}^{*}\boldsymbol{x}$ where $\boldsymbol{x}$ is the nonnegative eigenvector corresponding to $\rho((\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}})$. Then $\rho(\boldsymbol{J}_{\gamma,\omega})\leqslant\rho(\boldsymbol{J}_{\gamma,\omega}^{*})<1$.

Proof Since $\rho(\boldsymbol{L}_{\gamma,\omega})<1$, both $\breve{\boldsymbol{A}}=\breve{\boldsymbol{M}}-\breve{\boldsymbol{N}}$ and $\boldsymbol{A}^{*}=\boldsymbol{M}^{*}-\boldsymbol{N}^{*}$ are convergent splittings from Theorem 1. Thus, we only need to confirm that $\rho(\boldsymbol{J}_{\gamma,\omega})\leqslant\rho(\boldsymbol{J}_{\gamma,\omega}^{*})$. Since $(\breve{\boldsymbol{M}})^{-1}\geqslant(\boldsymbol{M}^{*})^{-1}$ and $\breve{\boldsymbol{A}}\boldsymbol{x}\geqslant\boldsymbol{A}^{*}\boldsymbol{x}$, we obtain

$$(\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{A}}\boldsymbol{x}-(\boldsymbol{M}^{*})^{-1}\boldsymbol{A}^{*}\boldsymbol{x}=(\boldsymbol{M}^{*})^{-1}\boldsymbol{N}^{*}\boldsymbol{x}-(\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}}x\geqslant 0,$$

from $(\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}}\boldsymbol{x}\geqslant\rho((\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}})\boldsymbol{x}$, the above inequality becomes $\rho((\breve{\boldsymbol{M}})^{-1}\breve{\boldsymbol{N}})\boldsymbol{x}\leqslant(\boldsymbol{M}^{*})^{-1}\boldsymbol{N}^{*}\boldsymbol{x}$. By Lemma 2, we have $\rho(\boldsymbol{J}_{\gamma,\omega})\leqslant\rho(\boldsymbol{J}_{\gamma,\omega}^{*})$.

Remark 5 Under the assumption in Theorem 4, the spectral radius of the iteration matrix of our preconditioned AOR iterative method is smaller than those methods with the preconditioners in [3,4,6,7].

Remark 6 Throughout this section, similar results about Jacobi, Gauss-Seidel, JOR and SOR methods can be obtained by choosing special parameters of ω and γ.

2 Numerical Example

Example 1 Give the two-dimensional convection-diffusion equation

$$-\Delta u+\frac{\partial u}{\partial x}+2\ \frac{\partial u}{\partial x}=f$$

in the unit square Ω with Dirichlet boundary conditions, if we apply the central difference scheme on a uniform grid with $N\times N$ interior nodes ($N^2=n$) to the discretization of the above equation, we obtain a system of linear equation (1) with the coefficient matrix

$$\boldsymbol{A}=\boldsymbol{I}\otimes\boldsymbol{P}+\boldsymbol{Q}\otimes\boldsymbol{I},$$

where $\boldsymbol{P}=\mathrm{tridiag}(-\frac{2+h}{8},1,-\frac{2-h}{8})$ and $Q=\mathrm{tridiag}(-\frac{1+h}{4},0,-\frac{1-h}{8})$ are $N\times N$ tridiagonal matrices, and $h=\frac{1}{N}$ is the step size.

Obviously, $\boldsymbol{A}$ is a sparse M-matrix. In our experiment, we choose zero vector as the initial iterative vector $\boldsymbol{x}^0$, and the right-hand-side vector is taken so that $\boldsymbol{e}=[1,1,\cdots,1]^{\mathrm{T}}$ is the solution of the considered system. The iterations are terminated when the norm of $\boldsymbol{x}^i-\boldsymbol{e}$ is less than 10^{-6}. In Table 1, we show briefly the basic iterative AOR method, the CPU time, the iteration number and the spectral radius with AOR, CPU, IT and ρ respectively. We take $\alpha_i=0.9$ for all i, let $\boldsymbol{P}^*$ AORI, $\boldsymbol{P}^*$ AORII represent the preconditioners $\boldsymbol{I}+\boldsymbol{S}_\alpha$ and $\boldsymbol{I}+\boldsymbol{S}_m$ respectively, and $\overset{\cup}{\boldsymbol{P}}$ AOR represent our new preconditioner $\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta$. In Table 2, $\boldsymbol{P}^*$ GMRESI and $\boldsymbol{P}^*$ GMRESII indicate preconditioned GMRES method with the preconditioners $\boldsymbol{I}+\boldsymbol{S}_\alpha$ and $\boldsymbol{I}+\boldsymbol{S}_m$, while $\overset{\cup}{\boldsymbol{P}}$ GMRES means the preconditioned GMRES method with the new preconditioner $\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta$. Our test was implemented on a PC using MATLAB programming package.

Table 1　IT and CPU for AOR and preconditioned AOR for Example 1

ω	γ	n	AOR			P^* AORI			P^* AORII			$\overset{\cup}{P}$AOR		
			CPU	IT	ρ	CPU	IT	ρ	CPU	IT	ρ	CPU	IT	ρ
1	1	16^2	0.032 0	47	0.702 4	0.032 0	29	0.560 2	0.031 0	27	0.538 6	0.031 0	23	0.470 4
GS	METHOD	24^2	0.110 0	52	0.716 4	0.219 0	31	0.578 3	0.203 0	29	0.557 5	0.202 0	25	0.490 4
		40^2	1.576 0	56	0.724 0	3.76	33	0.588 3	3.713 0	31	0.568 0	3.635 0	27	0.501 4
		50^2	5.335 0	57	0.725 6	13.775 0	34	0.590 5	13.759 0	32	0.570 2	13.494 0	27	0.503 8
1	0	16^2	0.032 0	95	0.838 1	0.032 0	72	0.789 2	0.093 0	70	0.783 1	0.078 0	62	0.750 4
JACOBI	METHOD	24^2	0.140 0	106	0.846 4	0.218 0	82	0.779 8	0.656 0	79	0.794 1	0.577 0	70	0.762 7
		40^2	1.8250	115	0.850 9	4.009 0	89	0.805 5	10.374 0	85	0.8	10.187 0	76	0.769 4
		50^2	5.975 0	117	0.851 8	14.320 0	90	0.806 7	14.196 0	88	0.801 3	14.102 0	78	0.770 8
0.75	0.75	16^2	0.063 0	81	0.818 8	0.062 0	55	0.748 3	0.062 0	54	0.738 7	0.047 0	47	0.700 4
SOR	METHOD	24^2	0.390 0	89	0.827 4	0.453 0	60	0.759 3	0.750 2	59	0.750 2	0.421 0	51	0.712 8
		40^2	5.819 0	95	0.832 2	7.690 0	64	0.765 4	7.675 0	62	0.756 4	7.628 0	54	0.719 6
		50^2	20.186 0	97	0.833 2	28.064 0	66	0.766 7	28.033 0	63	0.757 8	27.970 0	55	0.721 1
0.5	0.5	16^2	0.080 0	144	0.896 8	0.090 0	105	0.861 0	0.080 0	102	0.856 3	0.080 0	90	0.835 0
SOR	METHOD	24^2	0.620 0	159	0.901 9	0.690 0	115	0.867 4	0.700 0	111	0.863 0	0.640 0	99	0.842 4
		40^2	8.940 0	171	0.904 7	10.760 0	123	0.871 0	10.68	119	0.866 7	10.570 0	106	0.846 4
		50^2	33.555 0	174	0.905 2	41.681 0	126	0.871 7	39.265 0	121	0.867 4	38.486 0	108	0.847 3

From Table 1, we can see that $\overset{\cup}{\boldsymbol{P}}$ AOR is superior to $\boldsymbol{P}^*$ AORI and $\boldsymbol{P}^*$ AORII in computing time and iterative number, the spectral radius of $\overset{\cup}{\boldsymbol{P}}$ AOR is smaller than AOR, $\boldsymbol{P}^*$ AORI and $\boldsymbol{P}^*$ AORII, the computing time of AOR is less than $\overset{\cup}{\boldsymbol{P}}$ AOR whereas the iterative number of $\overset{\cup}{\boldsymbol{P}}$ AOR is less than AOR. Table 2 shows that $\overset{\cup}{\boldsymbol{P}}$ GMRES is superior to GMRES, $\boldsymbol{P}^*$ GMRESI and $\boldsymbol{P}^*$ GMRESII both in CPU time and iteration number.

In addition, we show the spectrum picture of matrix $\boldsymbol{A}$ in Fig. 1, the spectrum pictures of the preconditioned matrices $\boldsymbol{I}+\boldsymbol{S}_\alpha$, $\boldsymbol{I}+\boldsymbol{S}_m$ and $\boldsymbol{I}+\boldsymbol{S}_\alpha+\boldsymbol{S}_M+\boldsymbol{S}_\delta$ in Fig. 2-4. In the preconditioned cases of Fig. 2-4, we take $N=50$ and $\alpha_i=0.9$ for all i. It is easy to see the eigenvalues of the original matrix $\boldsymbol{A}$ in Fig. 1 is scattered between 0.1 to 1.9, the spectral distribution of our new preconditioned matrix in Fig. 4 is more clustered than those of the two others in Fig. 2-3. Clearly, the preconditioner proposed in this paper is more effective.

Table 2 IT and CPU for GMRES and preconditioned GMRES for Example 1

Method	$n=16\times16$		$n=24\times24$		$n=40\times40$		$n=50\times50$	
	CPU	IT	CPU	IT	CPU	IT	CPU	IT
GMRES	0.062 0	84	0.39	92	2.683 0	96	6.584 0	97
P^* GMRESI	0.062 0	57	0.265 0	62	1.841 0	65	4.399 0	65
P^* GMRESII	0.047 0	56	0.265 0	61	1.779 0	63	4.396 0	64
$\breve{P}$GMRES	0.046 0	49	0.218 0	53	1.560 0	55	3.822 0	56

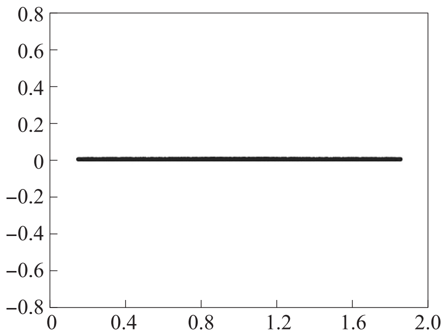

Fig. 1 Spectra of the matrix A

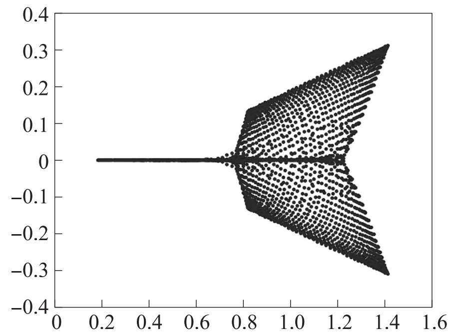

Fig. 2 Spectra of the matrix P^* AORI

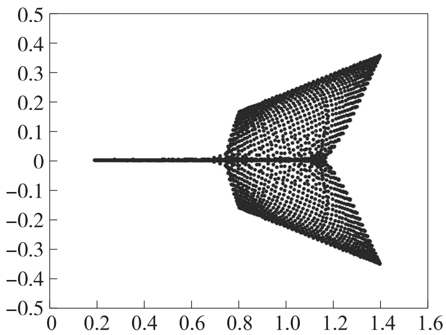

Fig. 3 Spectra of the matrix P^* AORII

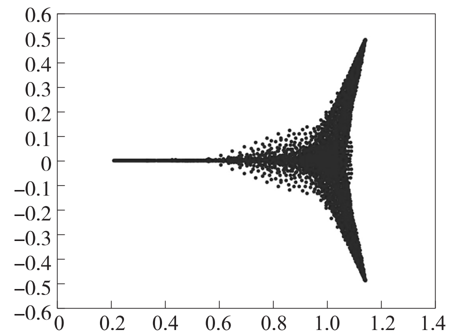

Fig. 4 Spectra of the new preconditioned matrix

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[参考文献]

[1] Hadjidimos A. Accelerated overrelaxation method[J]. Math Comput, 1978, 32: 149–157.

[2] Saad Y. Iterative Methods for Sparse Linear Systems[M]. Boston: PWS Publishing Company, 1996: 95–102.

[3] Gunawardena A D, Jain S K, Snyder L. Modified iterative methods for consistent linear systems[J]. Linear Algebra Appl, 1991, 154/156: 123–143.

[4] Kohno T, Kotakemori H, Niki H. Improving the modified gauss-seidel method for Z-matrix[J]. Linear Algebra Appl, 1997, 267: 113–123.

[5] Wu M, Wang L, Song Y. Preconditioned AOR iterative method for linear system[J]. Appl Numer Math, 2007, 57: 672–685.

[6] Kotakemori H, Harada K, Morimoto M, et al. A comparison theorem for the iterative method with the preconditoner $\boldsymbol{I}+\boldsymbol{S}_m$[J]. J Comput Appl Math, 2002, 145: 373–378.

[7] Morimoto M, Harada K, Sakakihara M, et al. The gauss-seidel iterative method with the preconditioned matrix $\boldsymbol{I}+\boldsymbol{S}+\boldsymbol{S}_m$[J]. Japan J Indust Appl Math, 2004, 21: 25–34.

[8] Berman A, Plemmons R J. Nonnegaitve Matrices in the Mathematical Sciences[M]. New York: Academic Press, 1979: 132–156.

[9] Varga R S. Matrix Iterative Analysis[M]. 2nd ed. Berlin: Springer-Verlag, 2000: 35–38.

[10] Wang L, Song Y. Preconditioned AOR iterative methods for M-matrices[J]. J Comput Appl Math, 2009, 226: 114–124.

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