A Fourier Pseudospectral Method for the Poisson Equation

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Abstract: In this paper, based on second-order Fourier spectral differentiation matrix $\boldsymbol{D}_2$ to approximate the second derivative, we obtain a standard Fourier pseudospectral full-discretization for the Poisson equation. According to the relationship between the spectral differentiation matrix and discrete Fourier transform, we provide a fast algorithm for solving the discrete equations. Some numerical results are presented. By using the FFT algorithm, numerical experiments show that the new scheme is very effective for calculation speed and easy to practice, and it has the high accuracy, these imply that the Fourier pseudospectral method provides a new useful tool for the study of the Poisson equation.

Key words: Fourier pseudospectral method, Poisson equation, FFT
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泊松方程的傅里叶拟谱方法

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[**摘要**]　本文基于二阶傅里叶拟谱微分矩阵来近似二阶导数,得到一个泊松方程的全离散傅里叶拟谱格式. 运用 FFT 理论分析了该数值格式,推导了快速方法,最后进行了数值试验. 数值试验显示数值方法求解速度快、方便实施,且高精度,说明该数值方法为泊松方程的研究提供了一个有效的新工具.

[**关键词**]　Fourier 拟谱,泊松方程,FFT

Many steady states of various physical phenomena can be described by a sort of so called elliptic partial differential equations. One can find its wide applications in heat-conducting problem, particle diffusion problem, the electrical current distribution in conductors, the electro-and magneto-statics problem, as well as in the elastic theory and permeation fluid model problem. Therefore, solving elliptic partial differential equation accurately is important and critical to precisely describing and understanding the relevant physical problems. The Poisson equation is one of the classical elliptic partial differential equations, which has been widely used in Plasma physics, plasma engineering. Sun[1] proposed a new 4th order accurate scheme combining the predictive and the corrective solutions. Ma[2] presented a new pseudo-spectral method which arranged the collocation points and the matrix of derivative for solving Poisson equation. Lü[3] analyzed the Poisson's equation from the aspect of structure. Poisson's equation is transformed to a Helmholtz-like equation which is solved iteratively, as a linear system of e-

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quations, by introducing a grid and employing fast Fourier transform by Ida M B Nielsen[4].

In this paper, we use the second-order Fourier spectral differentiation matrix $\boldsymbol{D}_2$ to approximate the second derivative, and obtain a standard Fourier pseudospectral discrete equation for the Poisson equation. We analyze the fast algorithm by FFT theory, and apply the FFT to solve the discrete equation. Some numerical results are present.

This paper is organized as follows. In Section 1, it analyzes construction of Fourier pseudospectral method for Poisson equation. And in Section 2, it analyzes the fast algorithm for Poisson equation. Finally, it has achieved good results after numerical tests.

1 Fourier Pseudospectral Method for Poisson Equation

We consider the Poisson equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y), \quad \text{in } \Omega, \tag{1}$$

$$u(x,y)=0, \quad \text{on } \partial\Omega. \tag{2}$$

Inspired by the spatial discrete method of Fourier pseudospectral, we discrete the solution through trigonometric polynomial at collocation points. So, we approximate $u(x,y)$ by $I_N u(x,y)$.

The approximation $I_N u(x,y)$ has the form

$$I_N u(x,y)=\sum_{l=0}^{N-1}\sum_{k=0}^{N-1} u_k^l \cdot g_k(x)\cdot g_l(y),$$

where $u_k^l=u(x_k,y_l)$, $g_k(x_i)=\delta_k^i$, $g_l(y_j)=\delta_l^j$, $x_i=\frac{L_x}{N}i$, $y_j=\frac{L_y}{N}j$, $i,j=0,1,2,\cdots,N-1$. Here N is an even number, L_x, L_y are the corresponding periods. $g_k(x)$ and $g_l(y)$ are trigonometric polynomials and given explicitly by

$$g_k(x)=\frac{1}{N}\sum_{n=-N/2}^{N/2}\frac{1}{c_n}\mathrm{e}^{\mathrm{i}n\mu_1(x-x_k)},$$

$$g_l(y)=\frac{1}{N}\sum_{n=-N/2}^{N/2}\frac{1}{c_n}\mathrm{e}^{\mathrm{i}n\mu_2(y-y_l)},$$

where $c_n=1\left(|n|\neq\frac{N}{2}\right)$, $c_{-\frac{N}{2}}=c_{\frac{N}{2}}=2$, $\mu_1=\frac{2\pi}{L_x}$, $\mu_2=\frac{2\pi}{L_y}$.

The derivative $\partial^k I_N u(x,y)/\partial x^k$ at the collocation points x_i in terms of the values u_i, the resulting expression at the points x_i is:

$$\frac{\partial^k I_N u(x_i,y_j)}{\partial x^k}=\sum_{m=0}^{N-1} u_m^j\frac{\mathrm{d}^k g_m(x_i)}{\mathrm{d}x^k}=(\boldsymbol{D}_k\boldsymbol{U})_{i,j},$$

where $\boldsymbol{D}_k$ is an $N\times N$ matrix with elements

$$(\boldsymbol{D}_k)_{i,m}=\frac{\mathrm{d}^k g_m(x_i)}{\mathrm{d}x^k}$$

and

$$\boldsymbol{U}=\begin{pmatrix} u_0^0 & u_0^1 & \cdots & u_0^{N-1}\\ u_1^0 & u_1^1 & \cdots & u_1^{N-1}\\ \vdots & \vdots & \ddots & \vdots\\ u_{N-1}^0 & u_{N-1}^1 & \cdots & u_{N-1}^{N-1}\end{pmatrix}.$$

In the same way, we can easily conclude

$$\frac{\partial^k I_N u(x_i,y_j)}{\partial y^k}=\sum_{n=0}^{N-1} u_i^n\frac{\mathrm{d}^k g_n(y_j)}{\mathrm{d}y^k}=(\boldsymbol{U}\overline{\boldsymbol{D}}_k^{\mathrm{T}})_{i,j},$$

where $\overline{\boldsymbol{D}}_k$ is an $N\times N$ matrix with elements

$$(\overline{\boldsymbol{D}}_k)_{j,n}=\frac{\mathrm{d}^k g_n(y_j)}{\mathrm{d}y^k}.$$

By using the spectral differentiation matrix $\boldsymbol{D}_2$, we obtain the standard Fourier pseudospectral full-discretization for the Poisson equation (1) as follows

$$\boldsymbol{D}_2\boldsymbol{U}+\boldsymbol{U}\overline{\boldsymbol{D}}_2^{\mathrm{T}}=\boldsymbol{F}, \tag{3}$$

here $\boldsymbol{F}=\begin{pmatrix} f(x_0,y_0) & f(x_0,y_1) & \cdots & f(x_0,y_{N-1}) \\ f(x_1,y_0) & f(x_1,y_1) & \cdots & f(x_1,y_{N-1}) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_{N-1},y_0) & f(x_{N-1},y_1) & \cdots & f(x_{N-1},y_{N-1}) \end{pmatrix}$.

2 Derivation of the Fast Algorithm for Poisson Equation

We first introduce a series of useful lemmas.

Lemma 1[5] Let $\boldsymbol{A}\in\boldsymbol{C}^{n\times n}$, $\boldsymbol{B}\in\boldsymbol{C}^{n\times n}$ and $\boldsymbol{D}\in\boldsymbol{C}^{n\times n}$, then the matrix equation

$$\boldsymbol{A}\boldsymbol{X}+\boldsymbol{X}\boldsymbol{B}=\boldsymbol{D}$$

is equivalent to the following vector form

$$\boldsymbol{Q}\cdot\boldsymbol{Y}=\boldsymbol{D},$$

where

$$\boldsymbol{Q}=\boldsymbol{I}_n\otimes\boldsymbol{A}+\boldsymbol{B}^{\mathrm{T}}\otimes\boldsymbol{I}_n,$$
$$\boldsymbol{Y}=(x_{11},x_{21},\cdots,x_{n1},x_{12},x_{22},\cdots,x_{n2},\cdots,x_{1n},x_{2n},\cdots,x_{nn}),$$
$$\boldsymbol{D}=(d_{11},d_{21},\cdots,d_{n1},d_{12},d_{22},\cdots,d_{n2},\cdots,d_{1n},d_{2n},\cdots,d_{nn}).$$

Lemma 2[6] Let

$$m_l=\begin{cases} il\mu, & l=0,1,\cdots,\dfrac{N}{2}, \\ i(l-N)\mu, & l=\dfrac{N}{2}+1,\cdots,N-1, \end{cases}$$

and $\boldsymbol{M}=\mathrm{diag}(m_0,m_1,\cdots,m_{N-1})$.

Then we have

$$A_k=\mathscr{F}^H\boldsymbol{M}^k\mathscr{F},$$

where $\mathscr{F}$ is discrete Fourier transform.

According to Lemma 1, the discrete equation (3) can also be written as a standard linear system,

$$(\boldsymbol{I}\otimes\boldsymbol{D}_2+\overline{\boldsymbol{D}}_2\otimes\boldsymbol{I})\boldsymbol{u}=\boldsymbol{f}, \tag{4}$$

where

$$\boldsymbol{u}=(u_0^0,u_1^0,\cdots,u_{N-1}^0,u_0^1,u_1^1,\cdots,u_{N-1}^1,\cdots,u_0^{N-1},u_1^{N-1},\cdots,u_{N-1}^{N-1}),$$
$$\boldsymbol{f}=(f_0^0,f_1^0,\cdots,f_{N-1}^0,f_0^1,f_1^1,\cdots,f_{N-1}^1,\cdots,f_0^{N-1},f_1^{N-1},\cdots,f_{N-1}^{N-1}).$$

Since $\boldsymbol{D}_2$ is a full matrix, a naive approach using Gauss elimination for Eq.(4) would cost $O(N^6)$ operations. However, this cost can be significantly reduced by using a discrete version of "separation of variables"—the matrix decomposition method[7], known as the matrix diagonalization method in the field of spectral methods[8,9].

According to Lemma 2, we have

$$\boldsymbol{D}_2=\mathscr{F}^H\boldsymbol{M}_2\mathscr{F},$$

then

$$\boldsymbol{M}_2=\mathscr{F}\boldsymbol{D}_2\mathscr{F}^H,$$

where $\mathscr{F}$ is discrete Fourier transform with elements $\mathscr{F}_{j,n}=\dfrac{1}{\sqrt{N}}W_N^{-jn}$, and $\boldsymbol{M}_2$ is a diagonal matrix.

Multiplying Eq.(3) from the left by $\mathscr{F}$ and from the right by $\mathscr{F}^H$, we find that

$$(\mathscr{F}\boldsymbol{D}_2\mathscr{F}^H)(\mathscr{F}\boldsymbol{U}\mathscr{F}^H)+(\mathscr{F}\boldsymbol{U}\mathscr{F}^H)(\mathscr{F}\overline{\boldsymbol{D}}_2^{\mathrm{T}}\mathscr{F}^H)=\mathscr{F}\boldsymbol{F}\mathscr{F}^H. \tag{5}$$

Let $\widetilde{\boldsymbol{U}}=\mathscr{F}\boldsymbol{U}\mathscr{F}^{H}$ and $\widetilde{\boldsymbol{F}}=\mathscr{F}\boldsymbol{F}\mathscr{F}^{H}$, then Eq.(5) becomes

$$\boldsymbol{M}_2\widetilde{\boldsymbol{U}}+\widetilde{\boldsymbol{U}}\overline{\boldsymbol{M}}_2^{\mathrm{T}}=\widetilde{\boldsymbol{F}}, \tag{6}$$

where gives

$$\widetilde{\boldsymbol{U}}_{i,j}=\frac{\widetilde{\boldsymbol{F}}_{i,j}}{(\boldsymbol{M}_2)_{ii}+(\overline{\boldsymbol{M}}_2)_{jj}},1\leqslant i,\ j\leqslant N-1. \tag{7}$$

3 Numerical Experiments

For the numerical experiments, we consider Poisson equation

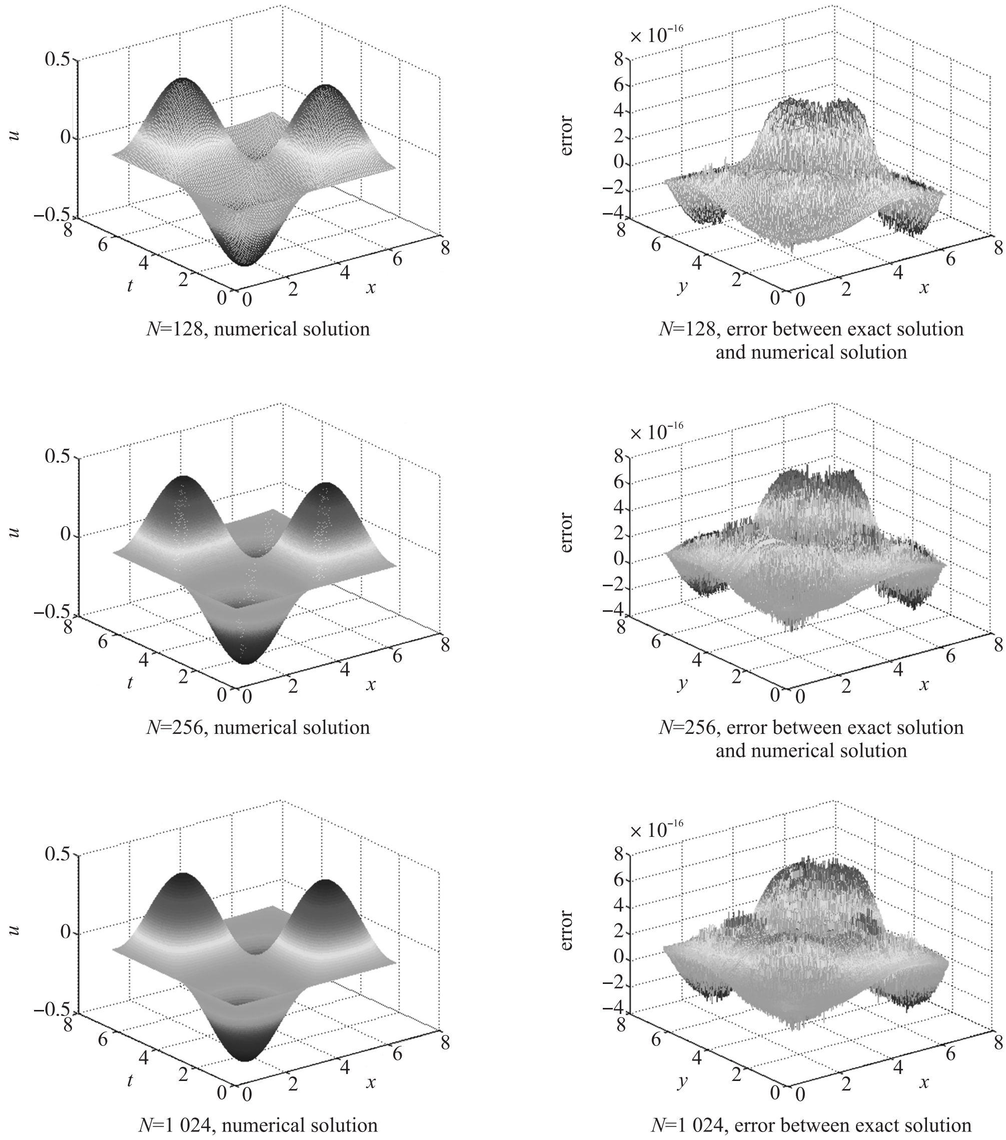
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\sin(x)\sin(y), \tag{8}$$

with the boundary condition $u(0,y)=u(2\pi,y)=0, u(x,0)=u(x,2\pi)=0$.

According to the analysis, we can obtain the full-discrete equation, i.e.,

$$\sum_{k=0}^{N-1}(\boldsymbol{D}_2)_{ik}u_k^j+\sum_{l=0}^{N-1}(\overline{\boldsymbol{D}}_2)_{jl}u_i^l=\sin(x_i)\sin(y_j). \tag{9}$$

We take $N=128,256$ and 1 024 for the new scheme (6), then we obtain the numerical solutions and the errors shown in Fig. 1, the exact solutions in Fig. 2.



N=128, numerical solution

N=128, error between exact solution and numerical solution

N=256, numerical solution

N=256, error between exact solution and numerical solution

N=1 024, numerical solution

N=1 024, error between exact solution and numerical solution

Fig. 1 Numerical solutions and errors

We consider that the problem is fast for solving the Poisson equation. Table 1 shows CPU time of the fast algorithm and traditional algorithm with different N. Table 1 clearly indicates that new scheme (6) not only can take large number for collocation points, but also has very fast calculation speed.

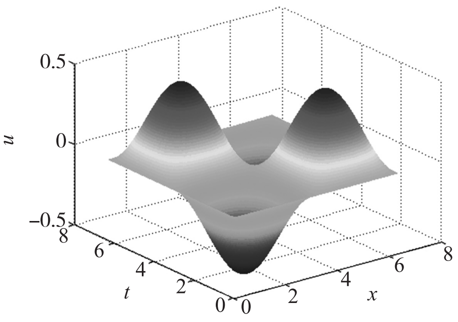


Fig. 2　Exact solutions

Table 1　CPU time of the fast algorithm and traditional algorithm with different N

N	CPU(s) for fast algorithm (6)	CPU(s) for traditional algorithm (4)
32	0.003 0	0.523 0
64	0.006 0	18.534 0
128	0.032 0	Out of memory
256	0.175 0	
512	1.908 0	
1 024	17.267 0	

4　Conclusions

The Fourier pseudospectral method provides a fast algorithm for the Poisson equation, numerical simulation results show its high accuracy. By using the FFT algorithm, numerical experiments show that the new scheme is very effective for calculation speed and easy to practice, and it has the high accuracy, therefore, it will be a good choice for solving the Poisson equation.

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