# $L^p$ Solutions of Backward Stochastic Differential Equations Driven by Fractional Brownian Motions

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**Abstract**: Recently, backward stochastic differential equations driven by fractional Brownian motion play an important role in mathematical finance, partial differential equations and other fields. In our paper, by the localization method and the generalized Ito formula, we consider the $L^p(p\geqslant 2)$ solutions of backward stochastic differential equations driven by fractional Brownian motions under reasonable assumptions.

**Key words**: backward stochastic differential equations, fractional Brownian motions, $L^p(p\geqslant 2)$ solutions, localization method

**CLC number**: 62M05, 60H10 **Document code**: A **Article ID**: 1001-4616(2015)04-0014-08

# 分数布朗运动驱动的倒向随机微分方程的$L^p$解

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[摘要] 分数次布朗运动驱动的倒向随机微分方程在金融数学、偏微分方程等领域有广泛应用.本文通过局部化方法以及推广的Ito公式,考虑了在一定条件下,分数布朗运动驱动的倒向随机微分方程中的$L^p$估计.

[关键词] 倒向随机微分方程,分数次布朗运动,$L^p(p\geqslant 2)$解,局部化方法

The nonlinear case of backward stochastic differential equations (BSDE, in Short) was first introduced by Pardoux and Peng[1]. It has the following form

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s)\mathrm{d}s-\int_t^T Z_s\mathrm{d}B_s, \tag{1}$$

where $B$ is a standard Brownian motion.

BSDE has wide applications in many research fields such as mathematical finance, stochastic control, non-linear analysis and so on. Since there are many models of physical phenomena which exploit the self-similarity and the long range dependence, fractional Brownian motion (fBm, in short) is a very useful tool to characterize such type of problems. Naturally, it is significant to study the BSDEs driven by fBms.

Let us recall that, an fBm $B^H=(B_t^H,t\geqslant 0)$ with Hurst parameter $H\in(0,1)$ is a mean zero Gaussian process whose covariance is given by

$$E[B^H(t)B^H(s)]=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}).$$

For $H=1/2$, the fBm is a standard Brownian motion. As was shown in [2], an fBm $B^H$ with Hurst parameter $H\neq 1/2$ is neither a semimartingale nor a Markov process. Thus, the classical stochastic theory cannot be

**Received data**: 2015-03-16.

**Foundation item**: Supported by the NSF of the Jiangsu Higher Education Committee of China(14KJB110016).

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used to define a stochastic integral with respect to fBm. In recent years, Duan and Hu (see e.g. [3-5]) have developed an efficient way to define stochastic integral with respect to fBm, which is based on Ito-Skorohod integral and fractional white noise.

BSDE driven by fBm (fBSDE, in short) was first introduced by Biagini et al.[6], when they studied the stochastic maximal principle in the framework of an fBm. There has been some literature on fBSDE. Hu and Peng[7] studies a general nonlinear fBSDE with the form

$$Y_t = g(\eta_T) + \int_t^T f(s,\eta_s,Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s^H, t \in [0,T] \ , \tag{2}$$

where $\eta_t = \eta_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H, \eta_0, b_s, \sigma_s$ are deterministic constants or functions, $g$ is a continuous function of polynomial growth and the generator $f$ satisfies a uniformly globally Lipschitz condition. They obtained the existence and uniqueness of the solutions under some mild assumptions by the quasi-conditional expectation of fBm. Based on [7], Maticiuc[8] is concerned with the existence and the uniqueness of the solution for a multi-dimensional fBSDE. Borkowska[9] proves the existence and uniqueness of generalized fBSDE which is driven by an additional term, i.e. an integral with respect to an increasing process. The BSDEs driven by both standard and fractional Brownian motions have been considered in [10]. Zhang[11] uses the quasi-conditional expectation to study the linear fBSDE. Moreover, the comparison theorem and the comonotonic theorem of the solution of linear BSDE are derived.

On the other hand, for BSDE driven by the standard Brownian motion, El Karoui, Peng and Quenez[12] first studies the $L^p(p>1)$ solution of problem (1) with the parameters also in the $L^p$ space. Many papers have been devoted to existence and uniqueness of $L^p$ solution of BSDE under weaker assumptions on the generator $f$. We can see the work of Briand [13] for a study of $L^p(p>1)$ solution with the generator $f$ satisfying monotonic condition, and the work of Chen [14] for a study of $L^p(1<p\leqslant 2)$ solution with continuous generator. In addition, Zhang and Zhao[15] prove the existence and uniqueness of the $L^p\times L^2(p\geqslant 2)$ solution of BSDE with $I$-growth coefficients. However, to our knowledge, there are few works on the $L^p(p\geqslant 2)$ solutions of fBDSE. Based on the existing works, our paper is to study the $L^p(p\geqslant 2)$ solutions of fBSDE (2) when $g(\eta_T)$ and $f(t,0,0,0)$ are in space. Here it needs to point out that the main problem of the study of BSDEs in the fractional framework is the absence of a martingale representation type theorem with respect to an fBm. In our approach, we use the localization method and the generalized Ito formula instead of constructing a contraction map on $(Y_t,Z_t)$ in $L^p(\Omega,F,\mathrm{P})$ space as shown in [12].

The rest of paper is organized as follows. In Section 2 we recall some definitions and results about fractional stochastic integrals. We study the existence and the uniqueness results of solution of fBSDE first, and then an equality about Malliavin derivative of solution is presented in Section 3. Section 4 is devoted to prove that the problem has a uniqueness solution $(Y_t,Z_t)$ for $Y_t \in L^p(\Omega,f,\mathrm{P})$ besides the $L^p$ norm estimates of solutions $(Y_t,Z_t)$ are obtained.

## 1 Prieliminaries

In this section, we recall some important definitions and results concerning the Ito-Skorohod integral with respect to an fBm, Ito formula and quasi-conditional expectation with respect to an fBm. For further details, the readers can refer to [16-17].

Throughout our paper, we assume that the Hurst parameter H always satisfies $H > 1/2$, Given continuous functions $\xi$ and $\eta$ on [0, T], we put

$$\langle \xi,\eta \rangle_t \triangleq \int_0^t\int_0^t \phi(u,v)\xi_u \eta_v \mathrm{d}u\mathrm{d}v \ ,$$

where $\phi(u,v)=H(2H-1)|u-v|^{2H-2}$. Settin $\xi=\eta$, we denote $\|\xi\|_t^2=\langle\xi,\xi\rangle_t$. For any $t\in[0,T]$, $\langle\xi,\eta\rangle_t$ is an Hilbert scalar product. Let $L_\phi^2([0,T])$ be the completion of the space of continuous functions under this scalar product, and we have the continuous embedding $L^2([0,T])\subset L_\phi^2([0,T])$ (see e.g.[18]).

Let $P_T$ be the set of elementary random variables of the form

$$F(\omega)=f(\int_0^T\xi_1(t)\mathrm{d}B_t^H,\dots,\int_0^T\xi_n(t)\mathrm{d}B_t^H),$$

where $f$ is a polynomial function of $n$ variables. The Malliavin derivative $D_s^H$ of an elementary variable $F\in P_T$ is defined by

$$D_s^HF=\sum_{j=1}^n\frac{\partial f}{\partial x_j}(\int_0^T\xi_1(t)\mathrm{d}B_t^H,\dots,\int_0^T\xi_n(t)\mathrm{d}B_t^H)\xi_j(s),\quad 0\leqslant s\leqslant T.$$

Consequently, we can further define another derivative by

$$\mathbb{D}_t^HF=\int_0^t\phi(t,v)D_v^HF\mathrm{d}v.$$

When the Hurst parameter $H>1/2$, we recall the following Ito formula for general integral process which has been proved in[4].

**Theorem 1** Let $\eta_t=\int_0^tF_u\mathrm{d}B_u^H,t\in[0,T]$, where $(F_t,0\leqslant t\leqslant T)$ is an $F_t$-adapted process in $L^2([0,T])$. Assume that there is an $\alpha>1-H$ such that $|u-v|\leqslant\delta,\delta>0$, such that $\lim\limits_{0\leqslant u,v\leqslant t,|u-v|\to0}E|\mathbb{D}_u^H(F_u-F_v)|^2=0$ and $E|F_u-F_v|^2\leqslant C|u-v|^{2\alpha}$. Let $f(t,x):[0,T]\times\mathbf{R}\to\mathbf{R}$ be a function having the continuous derivative in $t$ and the second order continuous derivative in $x$. Assume that the first order derivatives are bounded, and

$$E\int_0^T|F_S\mathbb{D}_s^H\eta_S|\mathrm{d}s<\infty,\left(f'(s,\eta_s)F_s,s\in[0,T]\right)\in L^2([0,T]).$$

Then

$$f(t,\eta_t)=f(0,0)+\int_0^t\frac{\partial f}{\partial s}(s,\eta_s)\mathrm{d}s+\int_0^t\frac{\partial f}{\partial x}(s,\eta_s)\mathrm{d}\eta_s+\int_0^t\frac{\partial^2f}{\partial x^2}(s,\eta_s)F_s\mathbb{D}_S^H\eta_s\mathrm{d}s,a.s.\quad t\in[0,T].$$

Next we will introduce the quasi-conditional expectation with respect to an fBm. For any $t\in(0,T]$, set $\boldsymbol{H}_t^{\otimes n}$ be the set of all real symmetric functions $f_n$ of $n$ variables on $[0,t]^n$ such that

$$\sum_{n=0}^\infty n!\int_{[0,t]^{2n}}\prod_{i=1}^n\phi(s_i,r_i)|f_n(s_1,\dots,s_n)||f_n(r_1,\dots,r_n)|\mathrm{d}s_1\cdots\mathrm{d}s_n\mathrm{d}r_1\cdots\mathrm{d}r_n<\infty.$$

Denote by $\hat{L}^2(\Omega,F,P)$ the set of $F\in L^2(\Omega,F,P)$ such that $F$ has the following chaos expansion

$$F=\sum_{n=0}^\infty\mathrm{I}_n(f_n).$$

For all $t\in[0,T]$, $f_n$, when restricted to $[0,t]^n$, is in $H_t^{\otimes n}$ and

$$I_n(f_n)\triangleq\int_{0\leqslant t_1,\dots,t_n\leqslant T}n!f_n(t_1,\dots,t_n)\mathrm{d}B_{t_1}^H\cdots\mathrm{d}B_{t_n}^H.$$

Then we recall the notation of quasi-conditional expectation and an important lemma which are stated in [7].

**Definition 1** If $F\in\hat{L}^2(\Omega,F,P)$ then the quasi-conditional expectation(see[17]) is defined as

$$\hat{E}[F|F_t]=\sum_{n=0}^\infty I_n\left(f_nI_{[0,t]}^{\otimes n}\right),$$

where

$$I_{[0,t]}^{\otimes n}(t_1,\dots,t_n)=I_{[0,T]}(t_1)\dots I_{[0,T]}(t_n).$$

**Theorem 2** If a real valued stochastic process $(F_u,0\leqslant u\leqslant T)$ satisfies

$$E\left\{\int_t^T\int_t^T\phi(u,v)|f_u||f_v|\mathrm{d}u\mathrm{d}v+\int_t^T\int_t^T\left|\mathbb{D}_u^Hf_v\right|^2\mathrm{d}u\mathrm{d}v\right\}<\infty$$

and $\int_t^Tf_u\mathrm{d}B_u^H\in\hat{L}^2(\Omega,F,P)$, then

$$\hat{E}\left[\int_t^T f_u \mathrm{d}B_u^H | F_t\right] = 0.$$

## 2 $L^2$ Solutions of fBDSE

In this section, we denote

$$\eta_t = \eta_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H, \quad 0 \leqslant t \leqslant T, \tag{3}$$

where $\eta_0$ is a constant, $b_t$ is a deterministic function in $t$ such that $\int_0^T |b(s)| \mathrm{d}s < \infty$, and $\sigma_t$ is a deterministic continuous function such that $\|\sigma\|_t$ exists for all $t$ with $\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma\|_t > 0$.

Now we consider the following fBSDE with the Hurst parameter $H > 1/2$:

$$\begin{cases} \mathrm{d}Y_t = -f\left(t, \eta_t Y_t, Z_t\right)\mathrm{d}t + Z_t \mathrm{d}B_t^H, \quad t \in [0, T], \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad Y_T = \xi, \end{cases} \tag{4}$$

where the generator $f(t, \eta, y, z) : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a progressively measurable function.

**Definition 2** A pair of $F_t-$ adapted processes $\left(Y_t, Z_t, 0 \leqslant t \leqslant T\right)$ is said to be a solution of fBSDE(4) if we have

$$Y_t = \xi + \int_t^T f\left(s, \eta_s, Y_S, Z_s\right)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s^H, \quad 0 \leqslant t \leqslant T.$$

Introduce the set

$$C_t = \{\phi(t, \eta_t) : \phi(t.x) \text{ is a continuously differentiable in } t \text{ and twice continuously differentiable in } x\}. \tag{5}$$

And denote by $V_T$ the set of process $Y_t$ with the form

$$V_T = \left\{Y_\cdot = \phi(\cdot, \eta_\cdot) : t \in [0, T], \phi(t, \eta_t) \in C_t\right\}. \tag{6}$$

Let $\bar{V}_T$ be the completion of $V_T$ under the following $\beta-$ norm

$$\|Y\|_\beta^2 = \int_0^T e^{\beta t} E|Y_t|^2 \mathrm{d}t = \int_0^T e^{\beta t} E|\phi(t, \eta_t)|^2 \mathrm{d}t. \tag{7}$$

Hung and Peng[7] first proves that a pair of solutions in $\bar{V}_T \times \bar{V}_T$ to BSDE(4) uniquely exists.

**Lemma 1** Given $\left(y_t, z_t\right) \in V_T \times V_T$, the following fBSDE with Hurst parameter $H > 1/2$,

$$\begin{cases} \mathrm{d}Y_t = -f\left(t, \eta_t y_t, z_t\right)\mathrm{d}t + Z_t \mathrm{d}B_t^H, \quad 0 \leqslant t \leqslant T. \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad Y_T = \xi, \end{cases} \tag{8}$$

has a solution $\left(Y_t, Z_t\right) \in V_T \times V_T$.

**Theorem 3** Let $\eta_t = \eta_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s^H$ be defined as(3). For some positive constant $c_0$, we assume that $\inf_{0 \leqslant t \leqslant T} \frac{\hat{\sigma}_t}{\sigma_t} \geqslant c_0$, where $\hat{\sigma}_t \triangleq \int_0^t \phi(t, r)\sigma_r \mathrm{d}r$. Let $\xi = g(\eta_T)$ be a square integrable martingale, where $g$ is a continuous function of polynomial growth. The generator $f : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is progressively measurable in $(y, z)$, $f(t, 0, 0, 0) \in L^2(\Omega, F, P)$ and $f$ satisfies uniformly globally Lipschitz condition with respect to $(y, z)$, i.e. there exists a positive constant $L > 0$ such that

$$|f(t, x, y, z) - f(t, x, y', z')| \leqslant L(|y - y'| + |z - z'|), \forall t \in [0, T], x, y, y', z, z' \in \mathbf{R}.$$

Then BSDE(4) has a unique solution $\left(Y_t, Z_t\right) \in \bar{V}_T \times \bar{V}_T$.

For the proof of Lemma 1 and Theorem 3, one can refer to Proposition 4.5 and Proposition 4.6 in [7] for details, respectively.

In order to obtain an equality about Malliavin derivative of solution $\left(Y_t, Z_t\right)$ in Theorem 3, we recall the following results.

**Lemma 2**(see [7]) Assume $b(s, x)$ and $a(s, x), 0 \leqslant s \leqslant T, x \in \mathbf{R}$, are continuous in $s$ and continuously differentiable in $x$ and both of them are of polynomial growth. For $\eta_t$ given as(3), if

$$\int_0^t b(s,\eta_s)\mathrm{d}s+\int_0^t a(s,\eta_s)\mathrm{d}B_s^H=0,\forall t\in[0,T].$$

Then

$$b(s,x)=a(s,x)=0,\quad \forall s\in[0,T],\quad x\in\mathbf{R}.$$

**Proposition 1** (see [7]) Assume that the fBSDE (4) has the solution with the form $(Y_t=u(t,\eta_t),Z_t=v(t,\eta_t),0\leqslant t\leqslant T)$, where $u(t,x)$ is continuously differentiable in $t$ and twice continuously differentiable in $x$. Then

$$v(t,x)=\sigma_t u_x(t,x).$$

Based on the above statements, the solution $\left(Y_t=u(t,\eta_t),Z_t=v(t,\eta_t),0\leqslant t\leqslant T\right)$ of (8) has the following property

$$D_t^H Y_t=\int_0^T\phi(t,r)D_r^H u(t,\eta_t)\mathrm{d}r=\int_0^T\phi(t,r)u_x(t,\eta_t)\sigma(r)1_{[0,t]}(r)\mathrm{d}r=\int_0^T\phi(t,r)\frac{Z_t}{\sigma_t}\sigma(r)1_{[0,t]}(r)\mathrm{d}r=\frac{\hat{\sigma}_t}{\sigma_t}Z_t.\tag{9}$$

We will show that the the solution $(Y_t,Z_t)\in\bar{V}_T\times\bar{V}_T$ has the same property as the form(9).

**Theorem 4** Assume that fBDSE(4) has a solution $(Y_t,Z_t)\in\bar{V}_T\times\bar{V}_T$, then for all $t\in[0,T]$, we have

$$D_t^H Y_t=\frac{\hat{\sigma}_t}{\sigma_t}Z_t.\tag{10}$$

**Proof** There exists a subsequence $(Y_k,Z_k)\in V_T\times V_T$ which converges to $(Y,Z)$ in $\bar{V}_T\times\bar{V}_T$ and satisfies

$$Y_{k+1}(t)=\xi+\int_t^T f\left(s,\eta(s),Y_k(s),Z_k(s)\right)\mathrm{d}s-\int_t^T Z_{k+1}(s)\mathrm{d}B_S^H,t\in[0,T].\tag{11}$$

By Proposition 1, $(Y_k,Z_k)$ has the form of

$$\left(Y_k(t)=u_k(t,\eta(t)),Z_k(t)=v_k(t,\eta(t))\right),t\in[0,T],$$

where $u_k,v_k\in C_t$ and

$$v_k(t,\eta(t))=\sigma(t)\frac{\partial u_k}{\partial x}(t,\eta(t)).$$

Moreover,

$$E\int_0^T e^{\beta t}|Y_k(t)-Y(t)|^2\mathrm{d}t\to 0$$

and

$$E\int_0^T e^{\beta t}|Z_k(t)-Z(t)|^2\mathrm{d}t\to 0.$$

Thus, for all almost $t\in[0,T]$, we have

$$\lim_{k\to\infty}E|Y_k(t)-Y(t)|^2=0$$

and

$$\lim_{k\to\infty}E|Z_k(t)-Z(t)|^2=0.$$

It follows that

$$D_r^H Y_k(t)=\frac{\partial u_k}{\partial x}(t,\eta(t))\sigma(r)1_{[0,t]}(r)=\frac{\sigma(r)}{\sigma(t)}Z_k(t)1_{[0,t]}(r)\to\frac{\sigma(r)}{\sigma(t)}Z(t)1_{[0,t]}(r),\tag{12}$$

as $k\to\infty$. Since $L^2([0,T])\subset L_\phi^2([0,T]),D_{r\,k}^H(t)$ also converges to

$$\frac{\sigma(r)}{\sigma(t)}Z_k(t)1_{[0,t]}(r)\ \text{ in }\ L^2(\Omega,F,P).$$

Furthermore, for all almost $t\in[0,T]$, we get

$$D_r^H Y(t)=\lim_{k\to\infty}D_r^H Y_k(t)=\lim_{k\to\infty}\frac{\sigma(r)}{\sigma(t)}Z_k(t)1_{[0,t]}(r)=\frac{\sigma(r)}{\sigma(t)}Z(t)1_{[0,t]}(r),\tag{13}$$

which implies

$$D_t^H Y_t = \int_0^T \phi(t,r) D_r^H Y(t) \mathrm{d}r = \frac{\hat{\sigma}_t}{\sigma_t} Z_t \tag{14}$$

and then completes the proof of Theorem 4.

## 3 $L^p$ Solutions of fBSDEs

In this section, we will study the $L^p$ solutions of fBSDE(4) based on the previous section.

**Theorem 5** We assume the conditions in Theorem 3 on $f, g, b_t, \sigma_t$. In addition, assume that both $b_t, \sigma_t$ are bounded, i.e. there exists a positive constant $M$ such that $|b_t| \leqslant M, |\sigma_t| \leqslant M$. The generator $f$ satisfies uniformly globally Lipschitz condition with respect to $(x,y,z)$ i. e. there exists a positive constant $L > 0$ such that

$$|f(t,x,y,z) - f(t,x',y',z')| \leqslant L(|x-x'| + |y-y'| + |z-z'|),$$

for any $t \in [0,T]$, and $x, x', y, y', z, z' \in \mathbf{R}$. Moreover,

$$E\int_0^T |f(s,0,0,0)|^{2p} \mathrm{d}s < \infty,$$

for $p \geqslant 1$. Then fBSDE(4) has a unique solution $(Y_t, Z_t, 0 \leqslant t \leqslant T)$ for $Y_t \in L^{2p}(\Omega, F, P)$ and $(Y_t, Z_t)$ satisfies the following inequalities

$$E\int_0^T |Y_t|^{2p} \mathrm{d}t \leqslant C\left(1 + E\int_0^T |f(s,0,0,0)|^{2p} \mathrm{d}s\right)$$

and

$$E\int_0^T |Y_t|^{2p-2} |Z_t|^2 \mathrm{d}t \leqslant C\left(1 + E\int_0^T |f(s,0,0,0)|^{2p} \mathrm{d}s\right),$$

where the constant $C$ depends only on $p, T, M, L, c_0$.

**Proof** By Theorem 3, fBSDE(4) has a unique solution $(Y_t, Z_t) \in \bar{V}_T \times \bar{V}_T$. For $M, N \in \mathbf{Z}^+$, set

$$\psi_M(y) = y^2 1_{\{-M \leqslant y < M\}} + M(2y - M) 1_{\{y \geqslant M\}} - M(2y + M) 1_{\{y < -M\}},$$
$$\varphi_{N,p}(y) = y^p 1_{\{0 \leqslant y < N\}} + N^{p-1}(py - (p-1)N) 1_{\{y \geqslant N\}}. \tag{15}$$

Applying Ito formula to the process $\psi_M(Y_t)$, we obtain

$$\mathrm{d}\psi_M(Y_t) = \psi'_M(Y_t)\mathrm{d}Y_t + \psi''_M(Y_t) Z_t D_t^H Y_t \mathrm{d}t. \tag{16}$$

By(16) and Theorem 4, we have

$$\mathrm{d}\psi_M(Y_t) = \left[-\psi'_M(Y_t) f(t, \eta_t, Y_t Z_t) + \psi''_M(Y_t) \frac{\hat{\sigma}_t}{\sigma_t} |Z_t|^2\right] \mathrm{d}t + \psi'_M(Y_t) Z_t \mathrm{d}B_t^H. \tag{17}$$

Furthermore, applying Ito formula to $\mathrm{e}^{\beta t} \varphi_{N,p}(\psi_M(Y_t))$, we deduce

$$\begin{aligned} \mathrm{d}\mathrm{e}^{\beta t} \varphi_{N,p}(\psi_M(Y_t)) = \beta \mathrm{e}^{\beta t} \varphi_{N,p}(\psi_M(Y_t)) \mathrm{d}t + \mathrm{e}^{\beta t} \varphi'_{N,p}(\psi_M(Y_t)) \mathrm{d}\psi_M(Y_t) + \mathrm{e}^{\beta t} \varphi''_{N,p}(\psi_M(Y_t)) \psi'_M(Y_t) Z_t D_t^H \psi_M(Y_t) \mathrm{d}t = \\ \beta \mathrm{e}^{\beta t} \varphi_{N,p}(\psi_M(Y_t)) \mathrm{d}t - \mathrm{e}^{\beta t} \varphi'_{N,p}(\psi_M(Y_t)) \psi'_M(Y_t) f(t, \eta_t, Y_t, Z_t) \mathrm{d}t + \mathrm{e}^{\beta t} \frac{\hat{\sigma}_t}{\sigma_t} \varphi'_{N,p}(\psi_M(Y_t)) \psi''_M(Y_t) |Z_t|^2 \mathrm{d}t + \\ \mathrm{e}^{\beta t} \varphi'_{N,p}(\psi_M(Y_t)) \psi'_M(Y_t) Z_t \mathrm{d}B_t^H + \mathrm{e}^{\beta t} \varphi''_{N,p}(\psi_M(Y_t)) \psi'_M(Y_t) Z_t D_t^H \psi_M(Y_t) \mathrm{d}t. \end{aligned} \tag{18}$$

In order to estimate the right hand side of(18), we first derive

$$\begin{aligned} D_t^H \psi_M(Y_t) = D_t^H Y_t^2 1_{\{-M \leqslant Y < M\}} + 2M D_t^H Y_t 1_{\{Y_t \geqslant M\}} - 2M D_t^H Y_t 1_{\{Y_t < -M\}} = 2\frac{\hat{\sigma}_t}{\sigma_t} Y_t Z_t 1_{\{-M \leqslant Y_t < M\}} + 2M \frac{\hat{\sigma}_t}{\sigma_t} Z_t 1_{\{Y_t \geqslant M\}} - \\ 2M \frac{\hat{\sigma}_t}{\sigma_t} Z_t 1_{\{Y_t < -M\}} = \frac{\hat{\sigma}_t}{\sigma_t} Z_t \left(2Y_t 1_{\{-M \leqslant Y_t < M\}} + 2M 1_{\{Y_t \geqslant M\}} - 2M 1_{\{Y_t < -M\}}\right) = \frac{\hat{\sigma}_t}{\sigma_t} Z_t \psi'_M(Y_t), \end{aligned} \tag{19}$$

where by the chain rule of Malliavin derivative the following equality is used

$$D_t^H Y_t^2 = \int_0^t \phi(t,r) D_r^H Y_t^2 \mathrm{d}r = \int_0^t \phi(t,r) 2Y_t D_t^H Y_t \mathrm{d}r = 2Y_t \int_0^t \phi(t,r) D_t^H Y_t \mathrm{d}r = 2Y_t D_t^H Y_t = 2\frac{\hat{\sigma}_t}{\sigma_t} Y_t Z_t. \tag{20}$$

Substituting(19) into(18) and integrating over $[t,T]$ yields

$$e^{\beta t}\varphi_{N,p}\big(\psi_M(Y_t)\big)=e^{\beta T}\varphi_{N,p}\Big(\psi_M\big(g(\eta_T)\big)\Big)-\beta\int_t^T e^{\beta s}\varphi_{N,p}\big(\psi_M(Y_s)\big)ds+\int_t^T e^{\beta s}\varphi'_{N,p}(\psi_M(Y_s))\psi'_M(Y_s)f(s,\eta_s,Y_s,Z_s)ds-$$
$$\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\varphi'_{N,p}(\psi_M(Y_s))\psi''_M(Y_s)\left|Z_s\right|^2 ds-\int_t^T e^{\beta s}\varphi'_{N,p}(\psi_M(Y_s))\psi'_M(Y_s)Z_s dB_s^H-$$
$$\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\varphi''_{N,p}(\psi_M(Y_s))\left|\psi'_M(Y_s)\right|^2\left|Z_s\right|^2 ds. \tag{21}$$

Noting that $(Y,Z)\in\bar{V}_T\times\bar{V}_T$ and $\varphi'_N\big(\psi_M(Y_t)\big),\psi'_M(Y_t)$ are bounded, we take the expectation on both sides of the equality(21)and get

$$e^{\beta t}E\varphi_{N,p}(\psi_M(Y_t))+\beta E\int_t^T e^{\beta s}\varphi_{N,p}(\psi_M(Y_s))ds+E\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\varphi''_{N,p}(\psi_M(Y_s))\left|\psi'_M(Y_s)\right|^2\left|Z_s\right|^2 ds+$$
$$E\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\varphi'_{N,p}(\psi_M(Y_s))\psi''_M(Y_s)\left|Z_s\right|^2 ds=e^{\beta T}E\varphi_{N,p}(\psi_M(g(\eta_T)))+E\int_t^T e^{\beta s}\varphi'_{N,p}(\psi_M(Y_s))\psi'_M(Y_s)f(s,\eta_s,Y_s,Z_s)ds. \tag{22}$$

According to the globally Lipschitz condition of $f$, we have

$$\left|f(s,\eta_s,Y_s,Z_s)\right|\leqslant\left|f(s,0,0,0)\right|+L\left|\eta_s\right|+L\left|Y_s\right|+L\left|Z_s\right|.$$

Thus it follows from(22) that

$$e^{\beta t}E\varphi_N(\psi_M(Y_t))+\beta E\int_t^T e^{\beta s}\varphi_N(\psi_M(Y_s))ds+E\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\varphi''_N(\psi_M(Y_s))\left|\psi'_M(Y_s)\right|^2\left|Z_s\right|^2 ds+$$
$$E\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\varphi'_N(\psi_M(Y_s))\psi''_M(Y_s)\left|Z_s\right|^2 ds\leqslant e^{\beta T}E\varphi_N(\psi_M(g(\eta_T)))+LE\int_t^T e^{\beta s}\left|\varphi'_N(\psi_M(Y_s))\right|\left|\psi'_M(Y_s)\right|\left|Y_s\right|ds+$$
$$LE\int_t^T e^{\beta s}\left|\varphi'_N(\psi_M(Y_s))\right|\left|\psi'_M(Y_s)\right|\left|Z_s\right|ds+LE\int_t^T e^{\beta s}\left|\varphi'_N(\psi_M(Y_s))\right|\left|\psi'_M(Y_s)\right|\left|\eta_s\right|ds+$$
$$E\int_t^T e^{\beta s}\left|\varphi'_N(\psi_M(Y_s))\right|\left|\psi'_M(Y_s)\right|\left|f(s,0,0,0)\right|ds. \tag{23}$$

By the monotonic convergence theorem, as $M\to\infty$ and $N\to\infty$ then we have

$$e^{\beta t}E\left|Y_t\right|^{2p}+\beta E\int_t^T e^{\beta s}\left|Y_s\right|^{2p}ds+(4p^2-2p)E\int_t^T e^{\beta s}\frac{\hat{\sigma}_s}{\sigma_s}\left|Y_s\right|^{2p-2}\left|Z_s\right|^2 ds\leqslant e^{\beta T}E|g(\eta_T)|^{2p}+2pLE\int_t^T e^{\beta s}|Y_s|^{2p}ds+$$
$$2pLE\int_t^T e^{\beta s}|Y_s|^{2p-2}|Y_s||Z_s|ds+2pLE\int_t^T e^{\beta s}|Y_s|^{2p-1}|\eta_s|ds+2pE\int_t^T e^{\beta s}|Y_s|^{2p-1}|f(s,0,0,0)|ds. \tag{24}$$

Then, by Young's inequality and the assumption that

$$\inf_{0\leqslant t\leqslant T}\frac{\hat{\sigma}_t}{\sigma_t}\geqslant c_0\,, \tag{25}$$

becomes

$$e^{\beta t}E|Y_t|^{2p}+\beta E\int_t^T e^{\beta s}|Y_s|^{2p}ds+\left(4p^2-3p\right)c_0E\int_t^T e^{\beta s}|Y_s|^{2p-2}|Z_s|^2 ds\leqslant e^{\beta T}E|g(\eta_T)|^{2p}+LE\int_t^T e^{\beta s}|\eta_s|^{2p}ds+$$
$$\left[\frac{pL^2}{c_0}+(2p-1)(L+1)\right]E\int_t^T e^{\beta s}|Y_s|^{2p}ds+E\int_t^T e^{\beta s}|f(s,0,0,0)|^{2p}ds, \tag{26}$$

where $\beta>\frac{pL^2}{c_0}+(2p-1)(L+1)$. Since both $b_s,\sigma_s$ are bounded, it is not difficult to check

$$LE\int_0^T e^{\beta s}|\eta_s|^{2p}ds<C_p. \tag{27}$$

On the other hand, $g$ is a continuous function of polynomial growth, which leads to

$$e^{\beta T}E|g(\eta_T)|^{2p}<C_p. \tag{28}$$

Combining(27)-(28)with(26), we arrive at

$$\left(\beta-\frac{pL^2}{c_0}-(2p-1)(L+1)\right)E\int_0^T e^{\beta s}|Y_s|^{2p}ds+\left(4p^2-3p\right)c_0E\int_0^T e^{\beta s}|Y_s|^{2p-2}|Z_s|^2 ds\leqslant C_p+E\int_0^T e^{\beta s}|f(s,0,0,0)|^{2p}ds. \tag{29}$$

which implies that $Y_t\in L^{2p}(\Omega,F,P)$ and the solution $(Y_t,Z_t)$ satisfies

$$E\int_0^T|Y_s|^{2p}ds<C\left(1+E\int_0^T|f(s,0,0,0)|^{2p}ds\right)$$

and

$$E\int_0^T |Y_s|^{2P-2}|Z_s|^2 ds < C\left(1 + E\int_0^T |f(s,0,0,0)|^{2p} ds\right).$$

Here the constant $C$ depends only on $p,T,M,L,c_0$. This puts an end of proof of Theorem 5.

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