Jacobi Sequences of $\sqrt{n^2 \pm 1}$

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Abstract: Let $p_k/q_k\,(k \geqslant 0)$ be the k th convergent of the continued fraction expansion of an irrational real number θ . We investigate the sequence of Jacobi symbols $(p_k/q_k)\,(k \geqslant 0)$. K. Girstmair showed that this sequence is purely periodic with period length 24 for $\theta = \mathrm{e}$ and period length 40 for $\theta = \mathrm{e}^2$. Similarly, in this paper, we determine the period lengths of the Jacobi sequences for $\theta = \sqrt{n^2+1}\,(n \geqslant 1)$ and $\theta = \sqrt{n^2-1}\,(n \geqslant 2)$.

Key words: continued fraction, Jacobi symbol, Jacobi sequence

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$\sqrt{n^2 \pm 1}$ 的雅可比序列

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[摘要] 令 $p_k/q_k\,(k \geqslant 0)$ 表示无理数 θ 的连分数展开式的第 k 个渐近分数.我们研究雅可比序列 $(p_k/q_k)\,(k \geqslant 0)$. K. Girstmair 证明了当 $\theta = \mathrm{e}$ 时,此序列是周期长度为24的纯周期序列;当 $\theta = \mathrm{e}^2$ 时,此序列是周期长度为40的纯周期序列.类似地,本文我们分别确定了 $\theta = \sqrt{n^2+1}\,(n \geqslant 1)$ 和 $\theta = \sqrt{n^2-1}\,(n \geqslant 2)$ 的雅可比序列的周期长度.

[关键词] 连分数,雅可比符号,雅可比序列

For any positive odd integer Q and any integer P with $\gcd(P,Q)=1$, (P/Q) is the Jacobi symbol[1-5]. If $P=0$, $Q=1$ or $P=1$, $Q=1$, we write $(P/Q)=1$. If Q is even and $\gcd(P,Q)=1$, then we put $(P/Q)=*$, where $*$ is an arbitrarily chosen symbol different from ± 1 [5]. Let $[a_0,a_1,a_2,\cdots]$ be the regular continued fraction expansion of an irrational real number θ and let $p_k/q_k\,(k \geqslant 0)$ be its k th convergent. The sequence (p_k/q_k) , $k \geqslant 0,$ is called the Jacobi sequence of θ .

K. Girstmair[3] proved that the Jacobi sequence of e is purely periodic with period length 24 and the sequence of e^2 is purely periodic with period length 40.

Similarly, in this paper, the following results are proved.

Theorem 1 The Jacobi sequence of $\sqrt{n^2+1}$, $n \geqslant 1$, is purely periodic with the smallest period length 8 if $2 \nmid n$ and 2 if $2 \mid n$.

Theorem 2 The Jacobi sequence of $\sqrt{n^2-1}$, $n \geqslant 2$, is purely periodic with the smallest period length 4 if $2 \nmid n$ and 16 if $2 \mid n$.

Remark 1 $\sqrt{n^2 \pm 1}$ are two kinds of irrational numbers that we are familiar with, and we know their continued fraction expansions explicitly. For any irrational real number θ , if its continued fraction expansion modulo 4 is period and has an explicit form, then we can obtain the similar result for the Jacobi sequence of θ . For example, we can obtain the parallel results for $\sqrt{7}$, $\dfrac{\sqrt{5}+1}{2}$, $\sqrt{n^2-n}=[n-1,\overline{2,2n-2}]\,(n \geqslant 2)$, $\sqrt{n^2-2}=[n-1,\overline{1,n-2,1,2n-2}]$ $(n \geqslant 3)$, $\sqrt{4n^2+4}=[2n,\overline{n,4n}]$, $\sqrt{(na)^2+a}=[na,\overline{2n,2na}]\,(a \in \mathbf{N})$, $\sqrt{(2n+1)^2+2n+1}=[2n+1,\overline{2,4n+2}]$ and so on.

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1 Lemmas

Lemma 1[1] Let $[\overline{a_0, a_1, \cdots, a_{L-1}}]$ be a purely periodic continued fraction with (not necessarily smallest possible) period length L and s_k/t_k be its k th convergent, $k \geqslant 0$. Supposed that L is even and

$$\left(\frac{t_{L-1}}{s_{L-1}}\right) = 1, \begin{pmatrix} s_{L-1} & s_{L-2} \\ t_{L-1} & t_{L-2} \end{pmatrix} \equiv \boldsymbol{I} (\mathrm{mod}\ 4), \tag{1}$$

where $\boldsymbol{I}$ is the unit matrix and congruence has to be understood entry by entry, then the Jacobi sequence of $[\overline{a_0, a_1, \cdots, a_{L-1}}]$ is purely periodic with period length L.

Lemma 2 Infinite periodic regular continued fractions $x = [a_0, \overline{a_1, \cdots, a_r}]$ and $y = [b_0, \overline{a_1, \cdots, a_r}]$ have the same Jacobi sequence.

Proof Let

$$\frac{p_n(x)}{q_n(x)}, \frac{p_n(y)}{q_n(y)}, \frac{p_n}{q_n}$$

denote n th convergents to continued fractions $[a_0, \overline{a_1, \cdots, a_r}]$, $[b_0, \overline{a_1, \cdots, a_r}]$ and $[\overline{a_1, \cdots, a_r}]$, respectively. Then

$$\frac{p_n(x)}{q_n(x)} = [a_0, a_1, \cdots, a_n] = a_0 + \frac{1}{[a_1, \cdots, a_n]} = a_0 + \frac{q_{n-1}}{p_{n-1}} = \frac{a_0 p_{n-1} + q_{n-1}}{p_{n-1}},$$

$$\frac{p_n(y)}{q_n(y)} = [b_0, a_1, \cdots, a_n] = b_0 + \frac{1}{[a_1, \cdots, a_n]} = b_0 + \frac{q_{n-1}}{p_{n-1}} = \frac{b_0 p_{n-1} + q_{n-1}}{p_{n-1}}.$$

Since $\gcd(p_{n-1}, q_{n-1}) = 1$, it follows that

$$(a_0 p_{n-1} + q_{n-1}, p_{n-1}) = 1, \quad (b_0 p_{n-1} + q_{n-1}, p_{n-1}) = 1.$$

Hence, for $n \geqslant 1$, we have the Jacobi symbols

$$\left(\frac{p_n(x)}{q_n(x)}\right) = \left(\frac{a_0 p_{n-1} + q_{n-1}}{p_{n-1}}\right) = \left(\frac{q_{n-1}}{p_{n-1}}\right)$$

and

$$\left(\frac{p_n(y)}{q_n(y)}\right) = \left(\frac{b_0 p_{n-1} + q_{n-1}}{p_{n-1}}\right) = \left(\frac{q_{n-1}}{p_{n-1}}\right).$$

For $n = 0$, we have

$$\left(\frac{p_n(x)}{q_n(x)}\right) = \left(\frac{a_0}{1}\right) = 1, \left(\frac{p_n(y)}{q_n(y)}\right) = \left(\frac{b_0}{1}\right) = 1.$$

Thus, we have proved that

$$\left(\frac{p_n(x)}{q_n(x)}\right) = \left(\frac{p_n(y)}{q_n(y)}\right)$$

for all $n \geqslant 0$. This completes the proof of Lemma 2.

Lemma 3 If $\alpha = [a_0, a_1, a_2, \cdots]$ and $\beta = [b_0, b_1, b_2, \cdots]$ are two infinite regular continued fractions such that $a_i \equiv b_i\ (\mathrm{mod}\ 4)$ for all $i \geqslant 1$, then α and β have the same Jacobi sequence.

Proof By [3, Theorem 2], we know that if $\alpha' = [a'_0, a'_1, a'_2, \cdots]$ and $\beta' = [b'_0, b'_1, b'_2, \cdots]$ are two regular continued fractions such that $a'_i \equiv b'_i\ (\mathrm{mod}\ 4)$ for all $i \geqslant 0$, then α' and β' have the same Jacobi sequence. Since $a_i \equiv b_i\ (\mathrm{mod}\ 4)$ for all $i \geqslant 1$, it follows that $\alpha = [a_0, a_1, a_2, \cdots]$ and $\beta_1 = [a_0, b_1, b_2, \cdots]$ have the same Jacobi sequence. By Lemma 2, $\beta_1 = [a_0, b_1, b_2, \cdots]$ and $\beta = [b_0, b_1, b_2, \cdots]$ have the same Jacobi sequence. Therefore, α and β have the same Jacobi sequence.

2 Proofs of Theorems

Proof of Theorem 1 By [4, p.321], we have $\sqrt{n^2+1} = [n, \overline{2n}]$, $n \geqslant 1$.

Case 1 n is even. Since $2n \equiv 4\ (\mathrm{mod}\ 4)$, it follows from Lemma 3 that $\sqrt{n^2+1}$ and $\sqrt{17} = [\overline{4}]$ have the same Jacobi sequence. Let s_k/t_k be the k th convergent of $\sqrt{17}$. Since

$$\left(\frac{t_1}{s_1}\right)=\left(\frac{4}{17}\right)=1,\begin{pmatrix} s_1 & s_0 \\ t_1 & t_0 \end{pmatrix}=\begin{pmatrix} 17 & 4 \\ 4 & 1 \end{pmatrix}\equiv \boldsymbol{I}(\bmod 4) ,$$

it follows from Lemma 1 with $L=2$ that the Jacobi sequence of $\sqrt{17}$ is purely periodic with period length 2. Since the Jacobi sequence of $\sqrt{17}$ starts with

$$1, *, 1, *, 1, *, \cdots ,$$

we know that the Jacobi sequence of $\sqrt{17}$ is purely periodic with the smallest period length 2. Therefore, the Jacobi sequence of $\sqrt{n^2+1}$ is purely periodic with the smallest period length 2.

Case 2 n is odd. Since $2n \equiv 2 \pmod 4$, it follows from Lemma 3 that $\sqrt{n^2+1}$ and $\sqrt{2}=[1,\overline{2}]$ have the same Jacobi sequence. By Lemma 2, $\sqrt{2}=[1,\overline{2}]$ and $[\overline{2}]$ have the same Jacobi sequence. Let s_k/t_k be the k th convergent of $[\overline{2}]$ and we choose $L=8$. Since

$$\left(\frac{t_7}{s_7}\right)=\left(\frac{408}{985}\right)=1,\begin{pmatrix} s_7 & s_6 \\ t_7 & t_6 \end{pmatrix}=\begin{pmatrix} 985 & 408 \\ 408 & 169 \end{pmatrix}\equiv \boldsymbol{I}(\bmod 4) ,$$

it follows from Lemma 1 that the Jacobi sequence of $[\overline{2}]$ is purely periodic with period length 8. Since the Jacobi sequence of $[\overline{2}]$ starts with

$$1, *, -1, *, -1, *, 1, *, 1, *, -1, *, -1, *, 1, *, \cdots ,$$

we know that the Jacobi sequence of $[\overline{2}]$ is purely periodic with the smallest period length 8. Therefore, the Jacobi sequence of $\sqrt{n^2+1}$ is purely periodic with the smallest period length 8.

Proof of Theorem 2 By [4, p.321], we have $\sqrt{n^2-1}=[n-1,\overline{1,2n-2}]$, $n \geqslant 2$.

Case 1 n is even. Since $2n-2 \equiv 2 \pmod 4$, it follows from Lemma 3 that $\sqrt{n^2-1}$ and $[1,\overline{1,2}]$ have the same Jacobi sequence.

The purely periodic number that is associated with $[1,\overline{1,2}]$ is $y=[\overline{1,2}]$. Let s_k/t_k be the k th convergent of y . Here, we choose $L=16$. Since

$$\left(\frac{t_{15}}{s_{15}}\right)=\left(\frac{21\,728}{29\,681}\right)=1,\begin{pmatrix} s_{15} & s_{14} \\ t_{15} & t_{14} \end{pmatrix}=\begin{pmatrix} 29\,681 & 10\,864 \\ 21\,728 & 7\,953 \end{pmatrix}\equiv \boldsymbol{I}(\bmod 4) ,$$

it follows from Lemma 1 that the Jacobi sequence of y is purely periodic with period length 16. We denote the sequence of convergents of $[1,\overline{1,2}]$ by

$$\frac{s'_0}{t'_0}, \frac{s'_1}{t'_1}, \frac{s'_2}{t'_2}, \cdots .$$

By [2, Theorem], we have

$$\left(\frac{s'_k}{t'_k}\right)=\left(\frac{s'_{k+16}}{t'_{k+16}}\right)$$

for all $k \geqslant 1$. Since

$$\left(\frac{s'_0}{t'_0}\right)=\left(\frac{s'_{16}}{t'_{16}}\right)=1$$

and the Jacobi sequence of $[1,\overline{1,2}]$ starts with

$$1, 1, -1, *, -1, -1, -1, *, -1, -1, 1, *, 1, 1, 1, *, 1, 1, \cdots ,$$

we know that the Jacobi sequence of $[1,\overline{1,2}]$ is purely periodic with the smallest period length 16. Therefore, the Jacobi sequence of $\sqrt{n^2-1}$ is purely periodic with the smallest period length 16.

Case 2 n is odd. Since $2n-2 \equiv 4 \pmod 4$, it follows from Lemma 3 that $\sqrt{n^2-1}$ and $[4,\overline{1,4}]$ have the same Jacobi sequence.

Similar to Case 1, we can prove that the Jacobi sequence of $[4,\overline{1,4}]$ is purely periodic with period length 8. Since the Jacobi sequence of $[4,\overline{1,4}]$ starts with

$$1,\ 1,\ 1,\ *,\ 1,\ 1,\ 1,\ *,\ 1,\ 1,\ 1,\ *,\ 1,\ 1,\ 1,\ *,\ \cdots,$$

the Jacobi sequence of $[4,\overline{1,4}]$ is purely periodic with the smallest period length 4. Therefore, the Jacobi sequence of $\sqrt{n^2-1}$ is purely periodic with the smallest period length 4.

3 Problems

Professor Yonggao Chen poses the following problems.

Problem 1 Are there an integer $L\geqslant 2$ and infinitely many primes p such that the Jacobi sequence of $\sqrt{p}$ is (purely) periodic with length L?

Problem 2 For any given integer $L\geqslant 2$, are there infinitely many primes p such that the Jacobi sequence of $\sqrt{p}$ is (purely) periodic with length L?

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