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On Dirichlet Problem of Tricomi-Type Equation in Rectangular Domains

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Abstract: Dirichlet problem of inhomogeneous Tricomi type equation in the rectangular domain $\Omega=\{(t_1,t_0)\times(0,\pi):t_1\leqslant 0, t_0>0\}$ is discussed. For $t_1=0$, we give the solution a priori estimate. For $t_1<0$, we show the Dirichlet problem is ill-posedness in Hadamard's sense by constructing a counterexample.

Key words: Tricomi-type equation, Dirichlet problem, well-posedness, ill-posedness

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Tricomi型方程在矩形区域上的Dirichlet问题

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[摘要] 讨论了非齐次Tricomi型方程在矩形区域$\Omega=\{(t_1,t_0)\times(0,\pi):t_1\leqslant 0,t_0>0\}$上的Dirichlet问题的适定性.当$t_1=0$时,建立了解的估计;当$t_1<0$时,构造反例说明Dirichlet问题在Hadamard's意义下是不适定的.

[关键词] Tricomi型方程,Dirichlet问题,适定性,不适定性

In this paper, we focus on the Dirichlet problem of inhomogeneous Tricomi type equation in the rectangular domain $\Omega=\{(t_1,t_0)\times(0,\pi):t_1\leqslant 0,t_0>0\}$. Namely, set m an odd positive integer, we consider

$$\begin{cases}\partial_t^2 u(t,x)-t^m\partial_x^2 u(t,x)=f(t,x),\\ u(t,0)=u(t,\pi)=0,\\ u(t_1,x)=\varphi(x),\\ u(t_0,x)=\psi(x).\end{cases}\tag{1}$$

When $t_1<0$, the equation (1) is a classic mixed-type equation, which is elliptic for $t\in(t_1,0)$ and is hyperbolic for $t\in(0,t_0)$, and $t=0$ is its degenerate line.

Tricomi equation ($m=1$ and $f(t,x)=0$) is a simplified model of transonic potential flow equation in hodograph plane, whose some boundary value problems may be lead to meaningful physical problems. For example, in consider of the stability of a complete rarefaction occurring on a concave body when a supersonic flow past, we should solve with Dirichlet problem of an equation with a local Tricomi type or Keldysh type degeneracy [1,2]. Meanwhile, the development of the theory on the well-posedness of the boundary value problem of mixed-type equation is fundamental in mathematics[3,4].

In general, the Dirichlet problem of hyperbolic equation is not well-posedness. Namely, its solution may neither exist, nor be uniquely determined, nor depend continuously on the data. But for degenerate hyperbolic equation there exist some positive results. In $\Omega_0=\{(0,t_0)\times(0,\pi):t_0>0\}$, (1) is a Dirichlet problem of a degenerate

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hyperbolic equation. In this case, there is a result for the uniqueness and existence of solution in $C(\Omega_{ABB_0A_0})\cap C^2(\Omega_{ABB_0A_0}\backslash(AB_0\cup BA_0))$ for the homogeneous equation in reference[1]if $\Omega_{ABB_0A_0}$ is an rectangular with two characteristics from opposite families AB_0: $x-\frac{2}{3}t^{\frac{3}{2}}=0$ and BA_0: $x+\frac{2}{3}t^{\frac{3}{2}}=1$. Later, for the equation

$$(-y)^m\partial_x^2 u-\partial_y^2 u-\lambda^2(-y)^m u=0,\ \text{ in } (0,1)\times(-\alpha,0)\ , \tag{2}$$

where $\alpha>0$, and

$$x=\frac{2}{m+2}(-y)^{\frac{m+2}{2}}\ \text{ and } 1-x=\frac{2}{m+2}(-y)^{\frac{m+2}{2}}$$

pass through the vertices of the domain, Khachev M M presented the unique solvability of the Dirichlet problem corresponding to(2)in[5]. Recently, there are some corresponding results on other mixed-type equation, see[6, 7]and so on. In this paper, we obtain the existence and uniqueness of the solution for the boundary value problems of Tricomi equation with non-zero source term by use of spectral analysis method in terms of confluent hypergeometric functions, meanwhile we release the constraints on the two characteristics which must pass through its vertices of the domain and obtain a higher regularity.

In a mixed-type domain, the most well-known example of the well-posedness boundary value problem for mixed-type equations is the Tricomi problem. Besides, Payne L E in[8]showed the ill-posedness of the initial-boundary value problem of Tricomi equation in Hadamard's sense by constructing a counterexample if the Cauchy data are given on the boundary in elliptic domain. For the Dirichlet problem, Morawetz C S proved the existence of the week solution for the Dirichlet problem of the Tricomi equation if the boundary defined in the elliptic domain is smooth and controlled by an algebraic function, and there is a sufficiently strong singularity at one of the two parabolic points of the boundary, the details can be found in[9]and more references can be refer to [10, 11]and the references therein. Here, for the rectangular domain $\Omega_1=\{(t_1,t_0)\times(0,\pi):\ t_1<0,t_0>0\}$, we give an example to show the ill-posedness of its Dirichlet problem in Hadamard's sense.

1 Main Result

Theorem 1 For $t_1=0$, prescribed $\varphi(x)$, $\psi(x)$ and $f(t,x)$ are third order continuously differential functions with respect to its variables. Moreover, given the conditions

$$\varphi(0)=\varphi(\pi)=\varphi''(0)=\varphi''(\pi)=0, \tag{3}$$

$$\psi(0)=\psi(\pi)=\psi''(0)=\psi''(\pi)=0, \tag{4}$$

$$f(t,0)=f(t,\pi)=\partial_x^2 f(t,0)=\partial_x^2 f(t,\pi)=0, \tag{5}$$

then there exists a unique smooth solution $C(\bar{\Omega}_0)\cap C^2(\Omega_0)$ of the problem(1), which satisfies the following estimate

$$\| u(t,x)\|_2\leqslant C(\|\varphi(x)\|_3+\|\psi(x)\|_3+\|f(t,x)\|_3), \tag{6}$$

where $\|u(x)\|_2=\sum_{i=0}^{2}\sup_{x\in\Omega_0}|D^i u(x)|$ and C is a positive constant.

Remark 1 The regularity condition on the Dirichlet data and the source term is not minimal for the existence of solution to the problem(1)when $t_1=0$. Here its regularity confirms the estimate(6).

Theorem 2 For $t_1<0$, the solution of the problem(1)in Ω_1 is unstable.

2 Proof of Theorem 1

In this section, we use the spectral analysis method and confluent hypergeometric functions to show Theorem 1.

The corresponding homogeneous Tricomi type equation of(1)is

$$\begin{cases}\partial_t^2 u(t,x)-t^m\partial_x^2 u(t,x)=0,\\ u(t,0)=u(t,\pi)=0,\\ u(t_1,x)=\varphi(x),\\ u(t_0,x)=\psi(x).\end{cases} \tag{7}$$

Setting $u(t,x)=T(t)X(x)$ and $\lambda>0$, we derive $\frac{T''(t)}{t^m T(t)}=\frac{X''(x)}{X(x)}=-\lambda$. Then combining with the boundary condition in the problem (1), it becomes

$$\begin{cases} X''(x)+\lambda X(x)=0, \quad 0\leqslant x\leqslant \pi, \\ X(0)=X(\pi)=0. \end{cases} \tag{8}$$

In terms of Sturm-Liouville theorem, its k-th eigenvalue $\lambda_k=k^2$ and the corresponding eigenfunction is $\sin(kx)$.

Assume the formal solution of the problem (1) as

$$u(t,x)=\sum_{k=1}^{\infty} T_k(t)\sin(kx) \tag{9}$$

and

$$\begin{cases} f(t,x)=\sum_{k=1}^{\infty} f_k(t)\sin(kx), \\ \varphi(x)=\sum_{k=1}^{\infty} \varphi_k\sin(kx), \\ \psi(x)=\sum_{k=1}^{\infty} \psi_k\sin(kx), \end{cases} \tag{10}$$

where $f_k(t)$, φ_k and ψ_k are the Fourier coefficients of $f(t,x)$, $\varphi(x)$ and $\psi(x)$ with respect to $\sin(kx)$ respectively. For the orthogonality and completeness of eigenfunctions, it is easy to verify the equation (1) is equivalent to

$$\begin{cases} T_k''(t)+k^2t^mT(t)=f_k(t), \\ T_k(0)=\varphi_k, \\ T_k(t_0)=\psi_k. \end{cases} \tag{11}$$

In order to solve the problem (11), we first consider the corresponding homogeneous equation

$$T_k''(t)+k^2t^mT(t)=0. \tag{12}$$

For $k=m=1$, this is the standard Airy equation. In order to solve with (12) for any $k\in\mathbf{N}^+$ and m, we set variable $y=\frac{2}{m+2}kt^{\frac{m+2}{2}}$ and introduce the new unknown function $Y(y)=T_k(t)$, then the equation (12) becomes into

$$yY''(y)+\frac{2}{m+2}Y'(y)+yY(y)=0. \tag{13}$$

If $y\neq 0$, i.e., $t\neq 0$, let variable $z=2\mathrm{i}y$ and introduce another new unknown function $Z(z)=\mathrm{e}^{\mathrm{i}y}Y(y)$, then it is easy to verify that (13) is equivalent to the confluent hypergeometric equation

$$zZ''(z)+\left(\frac{2}{m+2}\right)-z)Z'(z)-\frac{m}{2(m+2)}Z(z)=0. \tag{14}$$

In according to the formula (9) given in page 253 of [12], we derive the corresponding fundamental solutions of (12)

$$\begin{cases} T_{k1}(t)=\mathrm{e}^{-\frac{z}{2}}\Phi\left(\frac{m}{2(m+2)},\frac{m}{m+2},z\right), \\ T_{k2}(t)=t\mathrm{e}^{-\frac{z}{2}}\Phi\left(\frac{m+4}{2(m+2)},\frac{m+4}{m+2},z\right), \end{cases} \tag{15}$$

where $\Phi(a,c,z)$ is Humbert's symbol and analysis whose difinition can be found in page 271 of [12]. In order to analyze the properties of the above two functions, we cite the following result established in reference [13].

Lemma 1 Let a and c be real and $2a-c>-1$. Then any zero z_ν of the function $\Psi(a,c,z)$ must satisfy $\mathrm{Re}(z_\nu)<0$, where $\Psi(a,c,z)$ defined by

$$\Psi(a,c,z)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a,c,z)+\frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\Phi(a-c+1,2-c,z).$$

As a corollary of Lemma 1, for any $t>0$, $T_{kj}(t)\neq 0$, $j=1,2$ holds. Particularly, there exist $T_{kj}(t_0)\neq 0$. Meanwhile, in according to Lemma 3.1 in [14], the Wronskian determinant of $T_{k1}(t)$ and $T_{k2}(t)$ satisfy $W_k^{12}(t)=1$. Finally, we establish the general solution of (11):

$$T_k(t)=C_1T_{k1}(t)+C_2T_{k2}(t)+\int_0^t(T_{k2}(t)T_{k1}(\tau)-T_{k1}(t)T_{k2}(\tau))f_k(\tau)\mathrm{d}\tau$$

with $C_1=\varphi_k$ and

$$C_2=T_{k2}^{-1}(t_0)(\psi_k-\varphi_kT_{k1}(t_0)-\int_0^t(T_{k2}(t)T_{k1}(\tau)-T_{k1}(t)T_{k2}(\tau))f_k(\tau)\mathrm{d}\tau).$$

Therefore, we obtain the formal solution of (1) which can be written as

$$u(t,x)=\sum_{k=1}^{\infty}u_k(t,x)\ ,\tag{16}$$

where $u_k(t,x)=(A_{k1}(t)+A_{k2}(t)+A_{k3}(t)+A_{k4}(t))\sin(kx)$ and its coefficients expressed by

$$A_{k1}(t)=\varphi_kT_{k2}^{-1}(t_0)(T_{k2}(t_0)T_{k1}(t)-T_{k1}(t_0)T_{k2}(t))\ ,$$

$$A_{k2}(t)=\psi_kT_{k2}^{-1}(t_0)T_{k2}(t)\ ,$$

$$A_{k3}(t)=\int_0^t(T_{k2}(t)T_{k1}(\tau)-T_{k1}(t)T_{k2}(\tau))f_k(\tau)\mathrm{d}\tau\ ,$$

$$A_{k4}(t)=-T_{k2}^{-1}(t_0)\int_0^t(T_{k2}(t)T_{k1}(\tau)-T_{k1}(t)T_{k2}(\tau))f_k(\tau)\mathrm{d}\tau.$$

In order to show the series (16) is the solved solution, we prove the series (16) and its formally two order derivatives are uniformly convergent.

Lemma 2 For $-\pi<\arg z<\pi$ and large $|z|$, then

$$\Phi(a,c,z)=\frac{\Gamma(c)}{\Gamma(c-a)}(\mathrm{e}^{\mathrm{i}\pi\varepsilon}z^{-1})a\sum_{n=0}^{M}\frac{(a)_n(a-c+1)_n}{n\cdot(n-1)\cdots3\cdot2\cdot1}(-z)^{-n}+O(|z|^{-a-M-1})+$$
$$\frac{\Gamma(c)}{\Gamma(a)}\mathrm{e}^z z^{a-c}\sum_{n=0}^{N}\frac{(c-a)_n(1-a)_n}{n\cdot(n-1)\cdots3\cdot2\cdot1}z^{-n}+O(|\mathrm{e}^z z^{a-c-N-1}|)\ ,$$

where $\varepsilon=1$ if $\mathrm{Im}z>0$, $\varepsilon=-1$ if $\mathrm{Im}z<0$, $(a)_0\equiv1$, $(a)_n\equiv a(a+1)\cdots(a+n-1)$, and $M,N=0,1,2,\cdots$

First, the analyticity of $\Phi(a,c,z)$ implies that $T_{kj}(t)$, $j=1,2$ are bounded if $kt^{\frac{m+2}{2}}<C$. The asymptotic behavior of $\Phi(a,c,z)$ when z tending to infinity in Lemma 2 show $T_{kj}(t)$, $j=1,2$ are bounded if $kt^{\frac{m+2}{2}}>C$. Then, for any $t\in(0,t_0)$, $T_{kj}(t)$, $j=1,2$ are uniformly bounded with respect to k, i.e.,

$$|T_{kj}(t)|\leqslant C,\quad j=1,2.\tag{17}$$

Next, we derive the expressions of the derivatives of $T_{kj}(t)$. By direct computation with the expression (15) and formula $\frac{\mathrm{d}^n}{\mathrm{d}z^n}\Phi(a,c,z)=\frac{(a)_n}{(c)_n}\Phi(a+n,c+n,z)$ (see page 254 of [12]), one has

$$T_{k1}{}'(t)=\mathrm{i}kt^{\frac{m}{2}}\mathrm{e}^{-\frac{z}{2}}\left(\Phi\left(\frac{m}{2(m+2)}+1,\frac{m}{m+2}+1,z\right)-\Phi\left(\frac{m}{2(m+2)},\frac{m}{m+2},z\right)\right)=$$
$$\mathrm{i}\left(\frac{m+2}{4\mathrm{i}}\right)^{\frac{m}{m+2}}k^{\frac{2}{m+2}}z^{\frac{m}{m+2}}\mathrm{e}^{-\frac{z}{2}}\left(\Phi\left(\frac{3m+4}{2(m+2)},\frac{2(m+1)}{m+2},z\right)-\Phi\left(\frac{m}{2(m+2)},\frac{m}{m+2},z\right)\right)$$

and

$$T_{k2}{}'(t)=\mathrm{e}^{-\frac{z}{2}}\Phi\left(\frac{m+4}{2(m+2)},\frac{m+4}{m+2},z\right)+tz'(t)\partial_z\left(\mathrm{e}^{-\frac{z}{2}}\Phi\left(\frac{m+4}{2(m+2)},\frac{m+4}{m+2},z\right)\right)=$$
$$\mathrm{e}^{-\frac{z}{2}}\left(\Phi\left(\frac{m+4}{2(m+2)},\frac{2}{m+2},z\right)-\frac{(m+2)z}{4}\Phi\left(\frac{m+4}{2(m+2)},\frac{m+4}{m+2},z\right)\right).$$

Furthermore, we obtain the second order derivative of $T_{kj}(t)$, which satisfy

$$T_{k1}{}''(t)=2\left(\mathrm{i}\left(\frac{m+2}{4\mathrm{i}}\right)^{\frac{m}{m+2}}k^{\frac{2}{m+2}}\right)^2z^{\frac{m}{m+2}}\partial_z\left(z^{\frac{m}{m+2}}\mathrm{e}^{-\frac{1}{2}}\left(\Phi\left(\frac{3m+4}{2(m+2)},\frac{2(m+1)}{m+2},z\right)-\Phi\left(\frac{m}{2(m+2)},\frac{m}{m+2},z\right)\right)\right)=$$
$$2\left(\mathrm{i}\left(\frac{m+2}{4\mathrm{i}}\right)^{\frac{m}{m+2}}k^{\frac{2}{m+2}}\right)^2z^{\frac{2}{m+2}}\left(\left(\frac{m}{m+2}z^{-\frac{2}{m+2}}-\frac{1}{2}z^{\frac{2}{m+2}}\right)\mathrm{e}^{-\frac{z}{2}}\left(\Phi\left(\frac{3m+4}{2(m+2)},\frac{2(m+1)}{m+2},z\right)-\Phi\left(\frac{m}{2(m+2)},\frac{m}{m+2},z\right)\right)+\right.$$
$$\left.z^{\frac{m}{m+2}}\mathrm{e}^{-\frac{z}{2}}\frac{3m+4}{4(m+1)}\Phi\left(\frac{5m+8}{2(m+2)},\frac{3m+4}{m+2},z\right)-\frac{1}{2}\Phi\left(\frac{3m+4}{2(m+2)},\frac{2(m+1)}{m+2},z\right)\right)$$

and

$$T_{k2}''(t)=\mathrm{e}^{-\frac{z}{2}}\left(-\frac{1}{2}\Phi\left(\frac{m+4}{2(m+2)},\frac{2}{m+2},z\right)+\frac{(m+2)z}{8}\Phi\left(\frac{m+4}{2(m+2)},\frac{m+4}{m+2},z\right)+\frac{m+4}{4}\Phi\left(\frac{3m+8}{2(m+2)},\frac{m+4}{m+2},z\right)-\right.$$
$$\left.\mathrm{i}\left(\frac{m+2}{4\mathrm{i}}\right)^{\frac{m}{m+2}}\left(\frac{m+2}{4}\right)k^{\frac{2}{m+2}}z^{\frac{m}{m+2}}\Phi\left(\frac{m+4}{2(m+2)},\frac{m+4}{m+2},z\right)-\frac{m+2}{8}z\Phi\left(\frac{3m+8}{2(m+2)},\frac{2m+6}{m+2},z\right)\right).$$

Then, in terms of Lemma 2 and the expressions derived above, by the analyticity of the confluent hypergeometric function $\Phi(a,c,z)$, one has

$$|T_{kj}'(t)|\leqslant Ck^{\frac{m+4}{2(m+2)}}t^{\frac{m(m+1)}{2(m+2)}},\quad |T_{kj}''(t)|\leqslant Ck^{\frac{3m+8}{2(m+2)}}t^{\frac{3m}{2(m+2)}}. \tag{18}$$

Note that the functions $\varphi(x)$, $\psi(x)$ and $f(t,x)$ are third order continuously differential with respect to their variables, then integrating by parts with the conditions (3)-(5), one has

$$\begin{cases}\varphi_k=-\dfrac{1}{k^3}\displaystyle\int_0^{\pi}\varphi^{(3)}(x)\cos(kx)\mathrm{d}x,\\ \psi_k=-\dfrac{1}{k^3}\displaystyle\int_0^{\pi}\psi^{(3)}(x)\cos(kx)\mathrm{d}x,\\ f_k(t)=-\dfrac{1}{k^3}\displaystyle\int_0^{\pi}\partial_x^3 f(t,x)\cos(kx)\mathrm{d}x.\end{cases}$$

Hence, the estimates satisfy

$$|\varphi_k|\leqslant Ck^{-3}\|\varphi(x)\|_3,\ |\psi_k|\leqslant Ck^{-3}\|\psi(x)\|^3,\ |f_k(t)|\leqslant Ck^{-3}\|f(t,x)\|_3. \tag{19}$$

In terms of the above analysis (17)-(19), we conclude the following estimates of the coefficients in (16).

Lemma 3

$$\begin{aligned}
&|A_{k1}(t)|=O(k^{-3}\|\varphi(x)\|_3),\ |A_{k1}'(t)|=O(t^{\frac{m(m+1)}{2(m+2)}}k^{\frac{m+4}{2(m+2)}-3}\|\varphi(x)\|_3),\\
&|A_{k1}''(t)|=O(t^{\frac{3m}{2(m+2)}}k^{\frac{3m+8}{2(m+2)}-3}\|\varphi(x)\|_3);\\
&|A_{k2}(t)|=O(k^{-3}\|\psi(x)\|_3),\ |A_{k2}'(t)|=O(t^{\frac{m(m+1)}{2(m+2)}}k^{\frac{m+4}{2(m+2)}-3}\|\psi(x)\|_3),\\
&|A_{k2}''(t)|=O(t^{\frac{3m}{2(m+2)}}k^{\frac{3m+8}{2(m+2)}-3}\|\psi(x)\|_3);\\
&|A_{k3}(t)|=O(tk^{-3}\|f(t,x)\|_3),\ |A_{k3}'(t)|=O(t^{\frac{m^2+3m+4}{2(m+2)}}k^{\frac{m+4}{2(m+2)}-3}\|f(t,x)\|_3),\\
&|A_{k3}''(t)|=O(t^{\frac{5m+4}{2(m+2)}}k^{\frac{3m+8}{2(m+2)}-3}\|f(t,x)\|_3);\\
&|A_{k4}(t)|=O(k^{-3}\|f(t,x)\|_3),\ |A_{k4}'(t)|=O(t^{\frac{m(m+1)}{2(m+2)}}k^{\frac{m+4}{2(m+2)}-3}\|f(t,x)\|_3),\\
&|A_{k4}''(t)|=O(t^{\frac{3m}{2(m+2)}}k^{\frac{3m+8}{2(m+2)}-3}\|f(t,x)\|_3).
\end{aligned}$$

Based on Lemma 3, by direct computation with the estimates of the confluent hypergeometric functions, we derive

Lemma 4

$$\begin{aligned}
&|u_k(t,x)|\leqslant Ck^{-3}\left(\|\varphi(x)\|_3+\|\psi(x)\|_3+\|f(t,x)\|_3\right),\\
&|Du_k(t,x)|\leqslant Ck^{\frac{m+4}{2(m+2)}-3}\left(\|\varphi(x)\|_3+\|\psi(x)\|_3+\|f(t,x)\|_3\right),\\
&|D^2u_k(t,x)|\leqslant Ck^{\frac{3m+8}{2(m+2)}-3}\left(\|\varphi(x)\|_3+\|\psi(x)\|_3+\|f(t,x)\|_3\right).
\end{aligned}$$

Note that $\frac{m+4}{2(m+2)}<1$ and $\frac{3m+8}{2(m+2)}<2$ for positive odd number m, then the estimates given in Lemma 4 imply that the function defined in (9) is the C^2 solution of problem (1) and satisfies the estimate (6). Then, we complete the proof of Theorem 1.

3 Proof of Theorem 2

In this section, we construct a counterexample to show that if $\varphi(x)$ is prescribed on the line $t=t_1<0$, the other conditions are given as in Theorem 1, then the solution could not be guaranteed to small if the Dirichlet data is prescribed arbitrarily small.

Given the function

$$u(t,x)=v(t)\sin(kx), \tag{20}$$

where $v(t)$ is a solution of the ODE

$$v''(t)+k^2t^mv(t)=0. \tag{21}$$

Then the calculation in section 2 implies that $u(t,x)$ is a solution of the homogeneous Tricomi type equation

$$\partial_t^2u(t,x)-t^m\partial_x^2u(t,x)=0. \tag{22}$$

Consider a particular function,

$$v(t)=V(k,t)=\begin{cases} e^{-\frac{2ik}{m+2}t^{\frac{m+2}{2}}}\Psi\left(\dfrac{m}{2(m+2)},\dfrac{m}{m+2},\dfrac{4ik}{m+2}t^{\frac{m+2}{2}}\right), & t=iy^2, y>0, \\ e^{\frac{2ik}{m+2}t^{\frac{m+2}{2}}}\Psi\left(\dfrac{m}{2(m+2)},\dfrac{m}{m+2},\dfrac{4ik}{m+2}t^{\frac{m+2}{2}}\right), & t=iy^2, y<0, \end{cases} \tag{23}$$

where $\Psi(a,c,z)$ is defined in Lemma 1, whose asymptotic (page 278 of reference [12]) for large $|z|$ satisfies

$$\Psi(a,c,z)=\sum_{n=0}^{N}(-1)^n\frac{(a)_n(a-c+1)_n}{n\cdot(n-1)\cdots3\cdot2\cdot1}z^{-a-n}+O(|z|^{-a-N-1}) \tag{24}$$

with $N=0,1,2,\cdots$, $-\frac{3}{2}\pi<\arg z<\frac{3}{2}\pi$. By (24) for large k and fixed t_0, it is easy to verify

$$|v(t_0)|=|V(k,t_0)|=O(k^{-\frac{m}{2(m+2)}}). \tag{25}$$

Now we define

$$u(t,x)=V(k,t)\sin(kx). \tag{26}$$

By direct computation with the transformations used from (12) to (14), it is easy to verify that the function (26) satisfies (22) in $\Omega_1=\{(t_1,t_0)\times(0,\pi): t_1<0, t_0>0\}$ and the boundary conditions

$$u(t,0)=u(t,\pi)=0, t\in[t_1,t_0]$$

and for $x\in[0,\pi]$,

$$\varphi(x)=u(t_1,x)=V(k,t_1)\sin(kx), \tag{27}$$

$$\psi(x)=u(t_0,x)=V(k,t_0)\sin(kx). \tag{28}$$

Then, the functions $\varphi(x)$ and $\psi(x)$ can be made arbitrarily small on the boundaries $t=t_1$ and $t=t_0$ respectively as k become into positive infinity. But in the interior of the domain Ω_1, for $t=0$,

$$u(0,x)=\frac{\Gamma\left(\dfrac{m}{m+2}\right)}{\Gamma\left(\dfrac{m+4}{2(m+2)}\right)}\sin(kx). \tag{29}$$

Note that the function $u(t,x)$ defined in (26) is uniformly bounded in Ω_1 with respect to k. But (29) shows that $u(t,x)$ can arrive $\dfrac{\Gamma\left(\frac{m}{m+2}\right)}{\Gamma\left(\frac{m+4}{2(m+2)}\right)}$ in the interior of the domain Ω_1 no matter the smallness of the given data $\varphi(x)$ and $\psi(x)$. This imply that the solution is not continuously depended on the data, hence we conclude it unstable.

Finally, we complete the proof of Theorem 2.

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