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The Exact Domination Number of Generalized Petersen Graphs $P(n,k)$ with $n=3k$

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Abstract: A subset $S \subseteq V$ is a dominating set of $G=(V,E)$ if each vertex in $V \backslash S$ is adjacent to at least one vertex in S. The domination number of G is the cardinality of a minimum dominating set of G. Graph domination numbers and algorithms for finding them have been investigated for numerous classes of graphs, usually for graphs that have some kind of tree-like structure. In this paper, we determine that the exact domination number of generalized Petersen graphs $P(n,k)$ with $n=3k$, $\gamma(P(n,k))=\left\lceil \frac{5n}{9} \right\rceil$.

Key words: dominating set, domination number, generalized Petersen graph

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广义Petersen图的控制数($n=3k$)

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[摘要] 如果$V \backslash S$中的每一个点都与S中的至少一个点相邻,我们称V的子集S是$G=(V,E)$的一个控制集.G的控制数是G的最小控制集的基数.许多类型图的控制数及其算法已经被研究,通常这些图都有某种树型结构.本文将确定广义Petersen图当$n=3k$时的控制数,且其控制数为$\left\lceil \frac{5n}{9} \right\rceil$.

[关键词] 控制集,控制数,广义Petersen图

Let $G=(V,E)$ be a finite, undirected, simple graph. A subset $S \subseteq V$ is a dominating set of G if each vertex in $V \backslash S$ is adjacent to at least one vertex in S. The set V itself is such set.The domination number of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. A minimum dominating set of G is a $\gamma(G)$-set. A set S of vertices is effcient dominating set if each vertex of G is dominated by exactly one vertex in S. Domination number for graphs and associated consepts have been studied for many years and there is an extensive literature on the subject.In general, determining the domination number(and most of its variations)is an NP-compelet problem. For surveys on the domination concept in graph theory we refer the reader to[1,2].

The generalized Petersen graph $P(n,k)$ is the graph with vertex set $O \cup I$, where $O=\{O_1,O_2,\cdots,O_n\}$ and $I=\{I_1,I_2,\cdots,I_n\}$, and edge set $E_1 \cup E_2 \cup E_3$, where $E_1=\{O_iO_{i+1} | 1 \leqslant i \leqslant n\}$, $E_2=\{I_iI_{i+k} | 1 \leqslant i \leqslant n\}$ and $E_3=\{O_iI_i | 1 \leqslant i \leqslant n\}$. Here all the subscripts are to be read as integers modulo n. The domination number of generalized Petersen graphs has been studied[3-8]. In particular, Behzad et al[3] and Hong Yan et al[7] have determined the domination number of the generalized Petersen graphs $P(n,k)$ with $n=2k+1$, the exact domination number is $\left\lceil \frac{3n}{5} \right\rceil$. In this paper, we will determine the exact domination number of the generalized Petersen graphs $P(n,k)$

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with $n=3k$.

For convenience, we denote the generalized Petersen graphs $P(n,k)=P(3k,k)$ by $P(n)$ for any positive integer k in this section. In [4], the authors obtained the following useful results.

Lemma 1[4]

i) $\gamma(P(n,1))=\begin{cases}\frac{n}{2}+1 & \text{if } n\equiv 2 (\bmod 4)\\ \left\lceil\frac{n}{2}\right\rceil & \text{otherwise}\end{cases}$ for $n\geqslant 3$;

ii) $\gamma(P(n,2))=\left\lceil\frac{3n}{5}\right\rceil$ for $n\geqslant 5$;

iii) $\gamma(P(n,3))=\begin{cases}\frac{n}{2}+1 & \text{if } n\equiv 2 (\bmod 4)\\ \left\lceil\frac{n}{2}\right\rceil & \text{if } n\equiv 1,0\ (\bmod 4) \text{ or } n=11\\ \left\lceil\frac{n}{2}\right\rceil+1 & \text{if } n\equiv 3\ (\bmod 4) \text{ or } n\neq 11\end{cases}$ for $n\geqslant 7$;

iv) A generalized Petersen graph $P(n,k)$ has an efficient dominating set if and only if $n\equiv 0$ (mod 4) and k is odd.

By Lemma 1, we can easily obtain that $\gamma(P(3))=2$, $\gamma(P(6))=4$, $\gamma(P(9))=5$ and $\gamma(P(12))=7$.

In the following, we will discuss the exact value of $\gamma(P(n))$ for arbitrary $n=3k\geqslant 3$.

Theorem 1 If n is a positive integer such that $n=3k\geqslant 3$, then $\gamma(P(n))\leqslant\left\lceil\frac{5n}{9}\right\rceil$.

Proof Consider the sets $V(P(n))$, and $E(P(n))$ defined as above. Let $S_I:=\{I_t|1\leqslant t\leqslant k\}$, and $S_O:=\left\{O_{k+2+3(t-1)}|1\leqslant t\leqslant\left\lceil\frac{2k}{3}\right\rceil\right\}$. Then $S=S_O\cup S_I$ dominates $V(P(n))$ (see Fig.1).

Theorem 2 $\gamma(P(15))=9$.

Proof Let S be a $\gamma(P(15))-$ set of $P(15)$. By the definition of generalized Petersen graphs, $|V(P(15))|=30$ and the degree of each vertex is three, so each vertex can at most dominate four vertices including itself. Thus, $\gamma(P(15))\geqslant 8$. By Theorem 1, we know that $8\leqslant\gamma(P(15))\leqslant\left\lceil\frac{5\times 15}{9}\right\rceil=9$. Next we show that $|S|>8$, or equivalently that no 8 vertices of $P(15)$ form a dominating set. Suppose on the contrary that there is a dominating set D of $P(15)$ with $|D|=8$. Let $D_O=O\cap D$ and $D_I=I\cap D$, then $|D_O|+|D_I|=8$. We use the integer pair (i,j), where $i,j\in\{0,1,\cdots,8\}$ and $i+j=8$, to denote the situation that $|D_O|=i$ and $|D_I|=j$. We show that none of these situations would occur.

First, note that D_O dominates at most $3i$ vertices of the outer cycle $P[O]$, there are at least $15-3i$ vertices of O that need to be dominated by D_I, and each of them requires a dominator from I to dominate it, then we must have $|D_I|=8-i\geqslant 15-3i$, which implies $i\geqslant 4$ (since i is an integer). Similarly, D_I dominates at most $3j$ vertices of the inter cycle $P[I]$, then $|D_O|=8-j\geqslant 15-3j$, which implies $j\geqslant 4$. Thus, only the situation $(i,j)=(4,4)$ is possible to occur. In this situation, at most one vertex of $\{I_l,I_{l+5},I_{l+10}\}$ $(1\leqslant l\leqslant 5)$ is in D_I, and so there must exist three vertices $\{I_h,I_{h+5},I_{h+10}\}$ not in D $(h\in\{1,2,3,4,5\})$, without loss of generality, say I_1,I_6 and I_{11}, thus O_1,O_6 and O_{11} are in D. In order to dominate $\{O_3,O_4\}$, $\{O_8,O_9\}$ and $\{O_{13},O_{14}\}$, there must exist at least two vertices in D_O (see Fig.2), thus $|D_O|\geqslant 5$, a contradiction. Therefore, $\gamma(P(15))=9$.

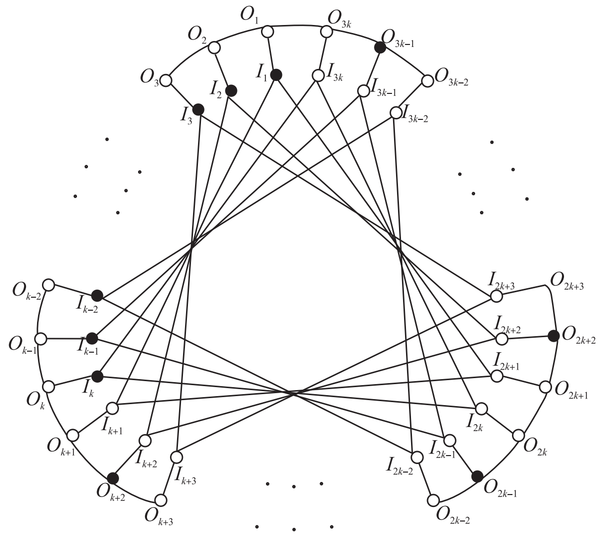

Fig.1　The case of $k=3q(q\in\mathbb{N})$

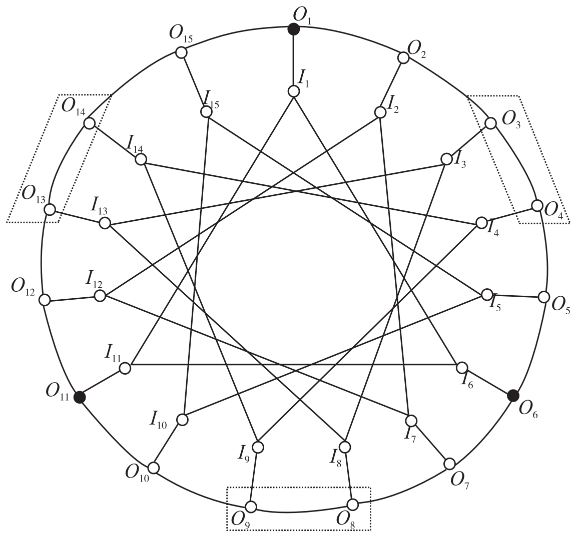

Fig.2　At least two vertices in D_o

Now we give an algorithm which constructs from $P(n)$ a smaller generalized Petersen graphs $P(n-9)$.

Algorithm

INPUT: the graph $P(n)=\left\{O\cup I, E_1\cup E_2\cup E_3\right\}$ with $n=3k\geqslant 18$.

OUTPUT: a graph P'' with $2(n-9)$ vertices.

Step 1　Choose i such that $1\leqslant i\leqslant k$, delete 24 vertices $M=\{O_j,I_j|i\leqslant j\leqslant i+3\}\cup\{O_j,I_j|i+k\leqslant j\leqslant i+k+3\}\cup\{O_j,I_j|i+2k\leqslant j\leqslant i+2k+3\}$ along with their 39 incident edges and denote the resulting graph by P'.

Step 2　Add 6 new vertices $N=\{O_i', I_i', O'_{i+k-3}, I'_{i+k-3}, O'_{i+2k-6}, I'_{i+2k-6}\}$, and define the graph P'' to have vertex set $V(P'')=V(P')\cup N$ and edge set $E(P'')=E(P')\cup\{O_{i-1}O_i', O_i'O_{i+4}, O_i'I_i', O_{i+k-1}O'_{i+k-3}, O'_{i+k-3}O_{i+k+4}, O'_{i+k-3}I'_{i+k-3}, I_i'I'_{i+k-3}, I'_{i+k-3}I'_{i+2k-6}, I'_{i+2k-6}I_i', O_{i+2k-1}O'_{i+2k-6}, O'_{i+2k-6}O_{i+2k+4}, O'_{i+2k-6}I'_{i+2k-6}\}$.

Return P''.

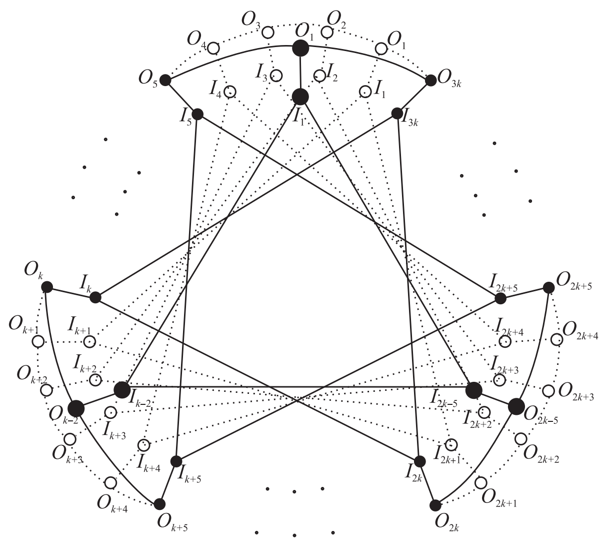

Fig.3　Algorithm for $i=1$

Lemma 2　For each a positive integer $n=3k\geqslant 18$, the graph P'' returned by Algorithm is isomorphic to $P(n-9)$.

Proof　It is clear that $|V(P'')|=2(n-9)$. Relabel the vertices of P'' as follows. For the chosen index i in step 1, set

$$U_i:=O_i', U_{i+k-3}:=O'_{i+k-3}, U_{i+2k-6}:=O'_{i+2k-6},$$

$$W_i := I_i', W_{i+k-3} := I'_{i+k-3}, W_{i+2k-6} := I'_{i+2k-6};$$

for each j such that $1 \leqslant j < i$, set

$$U_j := O_j, W_j := I_j;$$

for each j such that $i+4 \leqslant j < i+k$, set

$$U_{j-3} := O_j, W_{j-3} := I_j;$$

for each j such that $i+k+4 \leqslant j < i+2k$, set

$$U_{j-6} := O_j, W_{j-6} := I_j;$$

for each j such that $i+2k+4 \leqslant j \leqslant 3k = n$, set

$$U_{j-9} := O_j, W_{j-9} := I_j.$$

Then we get the sets $U = \{U_j | 1 \leqslant j \leqslant n-9\}$ and $W = \{W_j | 1 \leqslant j \leqslant n-9\}$ such that $V(P'') = U \cup W$. Note that $V(P(n-9))$ was defined to be $O \cup I$ with $|O| = |I| = n-9$, and the bijection $f: O \cup I \to U \cup W$, defined by $f(O_j) = U_j$ and $f(I_j) = W_j$ for $1 \leqslant j \leqslant n-9$, maintains adjacency and nonadjacency, the result follows immediately.

Theorem 3 Let n be a positive integer such that $n = 3k \geqslant 3$, then $\gamma(P(n)) \leqslant \gamma(P(n+9)) - 5$.

Proof To keep the notation in line with that of Algorithm, we may further assume that $n = 3k \geqslant 18$, and show that $\gamma(P(n-9) \leqslant \gamma(P(n)) - 5$. Let $P(n) = (O \cup I, E_1 \cup E_2 \cup E_3)$ be defined as before and S be a $\gamma(P(n))$-set of $P(n)$.

Let P'' be the graph returned by Algorithm with the index $i = 1$, then $P'' \cong P(n-9)$. We will identify $V(P(n-9))$ with $V(P'')$ such that $V(P(n-9)) = ((O \cup I) \backslash M') \cup N'$, where $M' = \{O_j, I_j | 1 \leqslant j \leqslant 4\} \cup \{O_j, I_j | k+1 \leqslant j \leqslant k+4\} \cup \{O_j, I_j | 2k+1 \leqslant j \leqslant 2k+4\}$ and $N' = \{O_1', I_1', O'_{k-2}, I'_{k-2}, O'_{2k-5}, I'_{2k-5}\}$. Let P' be the subgraph of $P(n)$, spanned by $V(P(n)) \backslash M'$, then P' is also a subgraph of $V(P(n-9))$, and the subset $S' = S \cap V(P')$ dominates all vertices in $V(P')$, except possibly vertices in $R = \{O_5, O_k, O_{k+5}, O_{2k}, O_{2k+5}, O_{3k}\}$. Let $Q = \{O_1, O_4, O_{k+1}, O_{k+4}, O_{2k+1}, O_{2k+4}\}$. We consider the following four cases.

Case 1 $|S \cap M'| \geqslant 8$.

Since $S'' = S' \cup \{O_1', O'_{k-2}, O'_{2k-5}\}$ forms a dominating set of P''. Thus, $\gamma(P(n-9)) = \gamma(P'') \leqslant |S''| = |S' \cup \{O_1', O'_{k-2}, O'_{2k-5}\}| \leqslant |S| - 5 = \gamma(P(n)) - 5$.

Case 2 $|S \cap M'| = 7$.

It is clear that $R \cap S' \neq \Phi$ (if $R \cap S' = \Phi$, then the subset of vertices contained in the closed dashed curve cannot be dominated by seven vertices, see Fig.4). Without loss of generality, say $O_5 \in R \cap S'$, then $S'' = S' \cup \{I'_{k-2}\}$ forms a dominating set of P''. Thus, $\gamma(P(n-9)) = \gamma(P'') \leqslant |S''| = |S' \cup \{I'_{k-2}, O'_{2k-5}\}| \leqslant |S| - 5 = \gamma(P(n)) - 5$.

Case 3 $|S \cap M'| = 6$.

By observation, we know that $|R \cap S'| \geqslant 2$ and $Q \cap S = \Phi$ (otherwise $|S \cap M'| > 6$). Now we consider two subcases.

Subcase 1 There are at least two pairs contain the vertices of S' in three pairs vertices $\{O_5, O_{3k}\}$, $\{O_k, O_{k+5}\}$ and $\{O_{2k}, O_{2k+5}\}$.

Without loss of generality, assume that O_5, $O_k \in R \cap S'$, then $S'' = S' \cup \{I'_{2k-5}\}$ forms a dominating set of P'' (see Fig.5). Thus $\gamma(P(n-9)) = \gamma(P'') \leqslant |S''| = |S' \cup \{I'_{2k-5}\}| \leqslant |S| - 5 = \gamma(P(n)) - 5$.

Subcase 2 Exactly one pair vertices of $\{\{O_5, O_{3k}\}, \{O_k, O_{k+5}\}, \{O_{2k}, O_{2k+5}\}\}$ are in S' and others are not in.

Without loss of generality, assume that O_5 and O_{3k} are in $R \cap S'$, O_k, O_{k+5}, O_{2k} and O_{2k+5} are not in $R \cap S'$. There is exact one vertex of $\{I_5, I_{k+5}, I_{2k+5}\}$ in S', then there are at least one vertex of $\{O_{k+6}, O_{2k+6}\}$ in S' and O_6 is not in S', (otherwise, if $O_6 \in S'$, let $S'' = (S' \backslash \{O_5\}) \cup \{O_4\}$, it can be considered as Case 2). In this situation, let P''' be the graph returned by Algorithm with the index $i = 2$, then $P''' \cong P(n-9)$. If O_{k+6} and O_{2k+6} are in S', then it can be considered by a similar argument with Subcase 1. If one of $\{O_{k+6}, O_{2k+6}\}$ is in S', without loss of generality, say $O_{2k+6} \in S'$ and $O_{k+6} \in S'$. Now, $O_1, O_6, O_{k+1}, O_{k+6}$ and O_{2k+1} are not in S', then the subset of vertices contained in the closed dashed curve can be dominated by at least seven vertices (see Fig.6), it can be considered by a similar argument with Case 1 or Case 2.

Case 4　$|S \cap M'| \leqslant 5$.

This case does not happen, since even if all vertices of R lie in S, the subset of vertices contained in the closed dashed curve cannot be dominated by five or fewer vertices (see Fig.7).

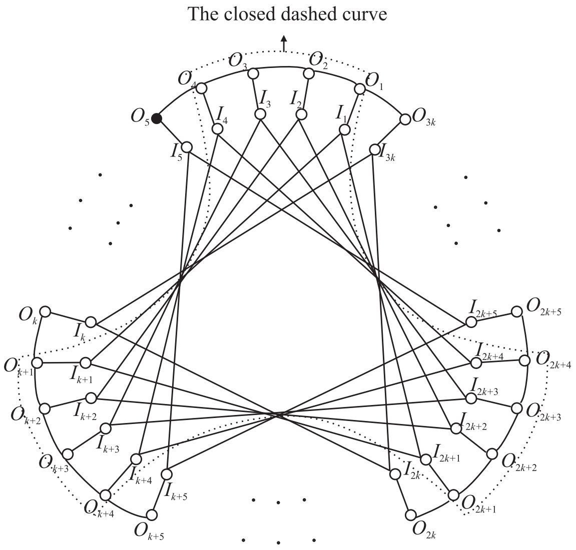

Fig.4　The mentioned vertices cannot be dominated by seven vertices

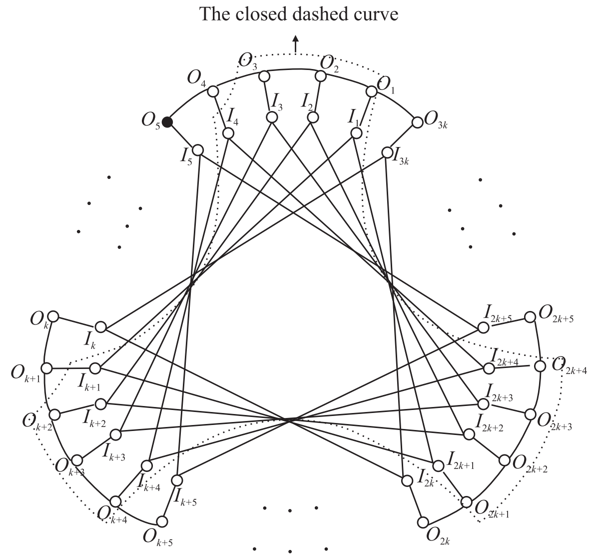

Fig.5　A dominating set of P''

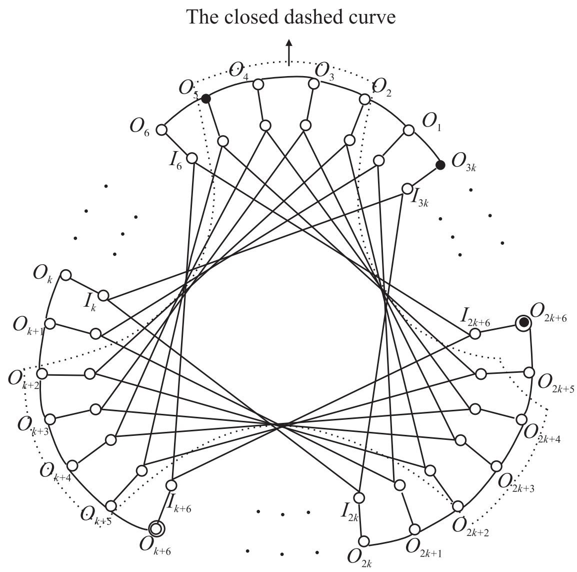

Fig.6　The mentioned vertices can be dominated by at least seven vertices

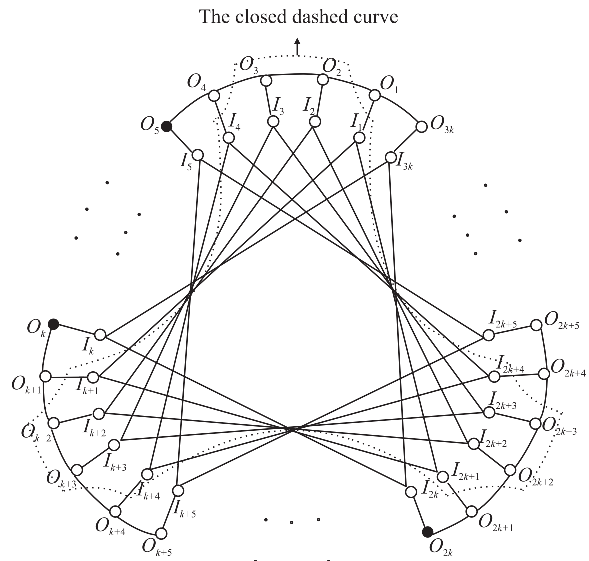

Fig.7　The mentioned vertices cannot be dominated by five or fewer vertices

Theorem 4　Let $P(n)$ be a generalized Petersen graphs with $n = 3k \geqslant 3$. The $\gamma(P(n)) = \left\lceil \frac{5n}{9} \right\rceil$.

Proof　By Theorem 1, we know that $\gamma(P(n)) \leqslant \left\lceil \frac{5n}{9} \right\rceil$. Now we prove that $\gamma(P(n)) \geqslant \left\lceil \frac{5n}{9} \right\rceil$. By contradiction. Define a graph class $\Omega = \left\{ P(n) \middle| \gamma(P(n)) < \left\lceil \frac{5n}{9} \right\rceil \right\}$. If $\Omega = \Phi$, we are done. Assume that $\Omega = \Phi$. Let $P(n) \in \Omega$ be the graph with minimum order $2n$. Then by Theorem 2, we have $n \geqslant 18$, and $\gamma(P(j)) = \left\lceil \frac{5j}{9} \right\rceil$ for each integer $j < n$. Consider the graph $P(n-9)$, by Theorem 3, we have $\gamma(P(n-9)) \leqslant \gamma(P(n)) - 5 < \left\lceil \frac{5n}{9} \right\rceil - 5 = \left\lceil \frac{5(n-9)}{9} \right\rceil$.

Hence we get a graph $P(n-9) \in \Omega$ with smaller order, which contradicts the choice of $P(n)$. Therefore we conclude that $\Omega = \Phi$, and the result holds.

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