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Directed Completions of Local Dcpos

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Abstract: In this paper, we explore directed completions of local dcpos. The following results are obtained. (1) The directed completions of continuous (resp., algebraic) local dcpos are continuous (resp., algebraic) dcpos; (2) The category CDcpo (resp., ADcpo) of continuous (resp., algebraic) dcpos and Scott continuous maps is a full reflective subcategory of the category CLDcpo (resp., ALDcpo) of continuous (resp., algebraic) local dcpos and local Scott continuous maps.

Key words: local dcpo, directed completion, continuous local dcpo

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局部定向完备集的定向完备化

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[摘要]　研究了局部定向完备集的定向完备化,得出以下两个结果:(1)连续(代数)的局部定向完备集的定向完备化是连续(代数)定向完备集;(2)连续(代数)定向完备集范畴是连续(代数)局部定向完备集范畴的满的反射子范畴.

[关键词]　局部定向完备集,定向完备化,连续的局部定向完备集

In the theoretical computer science, posets are required to be directed complete. However, there are important ordered sets such as the real sets **R** and the positive integers **N** which fail to be dcpos. Recent papers have contributed to the study of posets which are not directed complete. Xu Luoshan[1] introduced the concept of consistently continuous posets, and proved that the directed completions of consistently continuous posets are continuous posets, the category of continuous posets is a full reflective subcategory of consistently continuous posets. Guan and Wang[2] introduced the concept of continuous local dcpo. Moreover, Guan[2], Xu[3] inspected the properties of the Cartesian closed of relevant categories of local dcpos. In this paper, we prove that the directed completions of continuous local dcpos (resp., algebraic local dcpos) are continuous dcpos (resp., algebraic dcpos). In addition, we show that the category of continuous dcpos (resp., algebraic dcpos) and Scott continuous maps is a full reflective subcategory of the category of continuous local dcpos (resp., algebraic local dcpos) and local Scott continuous maps. The properties and characterizations of continuous local dcpo generalize the relevant results of continuous dcpo.

1　Preliminaries

We recall some basic notions and results about local dcpos. Let P be a poset. A subset $D\subseteq P$ is bounded directed if D is directed and has an upper bound in P. P is a local dcpo (in short, ldcpo) if $\downarrow p=\{x\in P: x\leqslant P\}$ is a

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dcpo for each $p\in P$. (It is different from the concept of consistently continuous posets, see Example 1). The notation $\bigvee_p D$ ($\bigvee_p^{\uparrow} D$) expresses a least upper bound of the (directed) subset $D\subseteq P$ in $\downarrow p$.

Let P be a ldcpo and $x,y\in P$. x is local way below y, denoted $x\ll_L y$, if for any directed subset $D\subseteq P$ and any upper bound p of D, if $y\leqslant \bigvee_p^{\uparrow} D$, then there exists some $d\in D$ with $x\leqslant d$. An element $k\in P$ is local compact if $k\ll_L k$, and let $K_L(P)$ be the subset of local compact elements of P. For each $p\in P$, let $\Downarrow_L p=\{x\in P: x\ll_L p\}$, if $\Downarrow_L p$ is directed and $p=\bigvee_p^{\uparrow}\Downarrow_L p$, then P is a continuous local dcpo (or briefly, cldcpo). P is an algebraic local dcpo (or briefly, aldcpo) if $\Downarrow_L p\cap K_L(P)$ is directed and $\bigvee_p^{\uparrow}\Downarrow_L p\cap K_L(P)=p$. An upper set U of P is local Scott open if $\bigvee_p^{\uparrow} D\in U$ implies $U\cap D\neq\varnothing$ for any bounded directed set $D\subseteq P$ and any upper bound p of D. The complement of a local Scott open set is called a local Scott closed set. The collection of all Scott open subsets of P is called the local Scott topology of P and is denoted by $\sigma_L(P)$. The collection of all Scott closed subsets of P is denoted by $\Gamma_L(P)$. A map $f: P\to Q$ between ldcpos is local Scott continuous if for any directed subset $D\subseteq P$ and any upper bound p of D, $f(\bigvee_p^{\uparrow} D)=\bigvee_p^{\uparrow} f(D)$ holds. It is known that f is local Scott continuous between ldcpos P and Q if and only if f it is continuous from the spaces $(P,\sigma_L(P))$ to $(Q,\sigma_L(Q))$. A non-empty local Scott closed set A is called a join-prime element if for any two local Scott closed sets B, C, the condition that $A\subseteq B\cup C$ implies either $A\subseteq B$ or $A\subseteq C$. Clearly, the closure of $\{x\}$ in the space $(Q,\sigma_L(Q))$ is a join-prime local Scott closed set. We use Spec $\Gamma_L(P)$ to denote the collection of all join-primes of $\Gamma_L(P)$.

Let P be a dcpo and $x,y\in P$. Obviously, P is also a cdcpo. It is easy to show that $x\ll_L y$ if and only if $x\ll y$ in P and $\sigma_L(P)=\sigma(P)$, where $\ll$ is the usual way below relation, $\sigma_L(P)$ is the usual Scott topology in P.

Example 1 Let $P=[0,1]\cup\{2\}$, where $[0,1]$ is the usual unit interval with the usual order and $x\leqslant 2$ for each $x\in P$. The elements 1, 2 are upper bounds of the directed set $[0,1)$ in P, but the set $[0,1)$ does not have a least upper bound in P. Then P is not a consistently continuous posets. However, $\downarrow x$ is a dcpo for each $x\in P$. Hence P is a ldcpo.

Lemma 1[2] Let P be a poset and $X\subseteq P$ with any upper bounds a, b. If $a\leqslant b$ or $b\leqslant a$, then $\bigvee_a X=\bigvee_b X$ (if $\bigvee_a X$, $\bigvee_b X$ exist).

Lemma 2[2] Let P be a ldcpo. For all $x,y\in P$, if $x\ll_L y$, then there exists a $z\in P$ such that $x\ll_L z\ll_L y$.

Lemma 3[2] Let P be a ldcpo. For each $x\in P$, $\Uparrow_L x$ is a local Scott open set in P.

Lemma 4[2] Let P be a ldcpo and $u,a,b,v\in P$. If $u\leqslant a\ll_L b\leqslant v$, then $u\ll_L v$.

Lemma 5[3] Let X be a lower subset of a ldcpo P. X is local Scott closed if for any bounded directed subset $D\subseteq X$ and any upper bound p of D, $\bigvee_p^{\uparrow} D\in X$.

2 Main Results

Similar to the proof of Proposition II-1.10[4], we have the following Proposition 1.

Proposition 1 Let P be a cldcpo. Then an upper set U is local Scott open if and only if for each $x\in U$ there is a $u\in U$ such that $u\ll_L x$.

Lemma 6 Let P be a cldcpo. If X be a lower set in P, then the set $\overline{X}=\{x\in P$: there is a bounded directed set $D\subseteq X$ with an upper bound b such that $x\leqslant\bigvee_b^{\uparrow} D\}$ is the closure of X in the space $(P,\sigma_L(P))$.

Proof Obviously, $\overline{X}$ is a lower set of P and $X\subseteq\overline{X}$. Let E be any directed subset of $\overline{X}$ and p any upper bound of E. Put $\bigvee_p^{\uparrow} E=e$. To verify that $\overline{X}$ is local Scott closed, it suffices to show $e\in\overline{X}$ by Lemma 5. Since P is a cldcpo, $\Downarrow_L e$ is directed and $e=\bigvee_e^{\uparrow}\Downarrow_L e$. For each $x\ll_L e$, there exists a $y\in P$ such that $x\ll_L y\ll_L e$ by Lemma 2. Then there exists a $v\in E$ such that $x\ll_L y\ll_L v$. Hence $v\in E\subseteq\overline{X}$, and there exists a bounded directed set $D\subseteq X$ with an upper bound b such that $v\leqslant\bigvee_b^{\uparrow} D$ by the definition of $\overline{X}$. So, there is a $d\in D$ such that $x\leqslant d$. Thus $x\in X$, since X is a lower set. Therefore, $\Downarrow_L e\subseteq X$ and $e\in\overline{X}$ by the definition of $\overline{X}$.

Now we show that $\overline{X}$ is the least local Scott closed set containing X. Suppose that B is local Scott closed and

$\overline{X}\subseteq B$. For each $x\in\overline{X}$, there exist a bounded directed subset $D\subseteq X$ and an upper bound b of D such that $x\leqslant\bigvee_b^{\uparrow}D$. Then $\bigvee_b^{\uparrow}D\in B$, since B is a local Scott closed set and $D\subseteq B$. Thus $x\in B$ and $\overline{X}\subseteq B$. Hence $\overline{X}$ is the closure of X in the space $(P,\sigma_L(P))$.

In this paper, for a cldcpo P, the symbol $\overline{X}$ expresses the closure of subset $X\subseteq P$ in the space $(P,\sigma_L(P))$. For each member p of a ldcpo P, $\downarrow p$ is the smallest lower set containing p. Then we have the following corollary by Lemma 6.

Corollary 1 Let P be a cldcpo. Then for each $p\in P$, $\overline{\{p\}}=\overline{\downarrow p}=\{x\in P$: there is a bounded directed set $D\subseteq\downarrow p$ with an upper bound b such that $x\leqslant\bigvee_b^{\uparrow}D\}$.

Proposition 2 Let P be a cldcpo. Spec $\Gamma_L(P)$ is closed under directed suprema in $\Gamma_L(P)$.

Proof Let Ω be a directed set of Spec $\Gamma_L(P)$. Then $\sup\Omega=\overline{\cup\Omega}$, where $\sup\Omega$ is the suprema of Ω in $\Gamma_L(P)$. Next, we show that $\sup\Omega\in$ Spec $\Gamma_L(P)$. Let F_1,F_2 be local Scott closed in P and $\sup\Omega\subseteq F_1\cup F_2$. We claim that $\cup\Omega\subseteq F_1$ or $\cup\Omega\subseteq F_2$. If not, there exist $D_1,D_2\in\Omega$ such that $D_1\subseteq F_1, D_1\not\subseteq F_2$ and $D_2\subseteq F_2, D_2\not\subseteq F_1$. Since Ω is directed, there is a $D_3\subseteq\Omega$ such that $D_1\subseteq D_3, D_2\subseteq D_3$. In addition, $D_3\subseteq F_1$ or $D_3\subseteq F_2$, since $D_3\in$ Spec $\Gamma_L(P)$ and $D_3\subseteq F_1\cup F_2$. Then $D_1\subseteq F_2$ or $D_2\subseteq F_1$. We get a contradiction. The proof is completed.

Lemma 7 Let D be a directed subset of a cldcpo P. Then $\overline{D}=\sup\{\overline{\{d\}}:d\in D\}\in$ Spec $\Gamma_L(P)$. Moreover, if D is a bounded directed subset of P and b is any upper bound of D, then $\overline{D}=\overline{\downarrow\bigvee_b^{\uparrow}D}=\overline{\{\bigvee_b^{\uparrow}D\}}$.

Proof Assume that D is a directed set of P. Then $\{\overline{\{d\}}:d\in D\}$ is directed in Spec $\Gamma_L(P)$. Thus $\sup\{\overline{\{d\}}:d\in D\}=\overline{\cup\{\overline{\{d\}}:d\in D\}}\in$ Spec $\Gamma_L(P)$ by Proposition 2. For each $d\in D$, we have $\overline{\{d\}}\subseteq\overline{D}$. Then $\cup\{\overline{\{d\}}:d\in D\}\subseteq\overline{D}$. Hence $\overline{\cup\{\overline{\{d\}}:d\in D\}}\subseteq\overline{D}$. On the other hand, since $D\subseteq\{\overline{\{d\}}:d\in D\}$, $\overline{D}\subseteq\overline{\{\overline{\{d\}}:d\in D\}}$. Hence

$$\overline{D}=\sup\{\overline{\{d\}}:d\in D\}=\overline{\cup\{\overline{\{d\}}:d\in D\}}\in\text{Spec }\Gamma_L(P).$$

Similarly, the case that D is bounded directed by Lemma 6 and Corollary 1.

By Proposition 2, if P is a cldcpo, then Spec $\Gamma_L(P)$ is directed under inclusion. Thus we have the following Definition 1.

Definition 1 Let P be a cldcpo. Spec $\Gamma_L(P)$ is called a directed completion of P.

Example 2 Let P be the ldcpo in Example 1. Clearly, P is a cldcpo, and Spec $\Gamma_L(P)\cong[0,1]$; Let $\mathbb{Z}$ be the set of integers with the usual order. Clearly, $\mathbb{Z}$ is an aldcpo, and Spec $\Gamma_L(\mathbb{Z})\cong\mathbb{Z}\cup\{+\infty\}$.

Definition 2 Let P be a cldcpo. For each $A\in$ Spec $\Gamma_L(P)$, put $A^*=\{b\in A$: there is an $a\in A$ such that $b\ll_L a\}$.

In Definition 2, it is clear that $A^*\subseteq A$ by the definition of A^*.

Lemma 8 Let P be a cldcpo. If $A\in$ Spec $\Gamma_L(P)$, then

(1) A^* is a directed lower set in P; (2) $A=\overline{A^*}=\sup\{\overline{\{b\}}:b\in A^*\}$; (3) For all $B,C\in\Gamma_L(P)$, $B\subseteq C$ if and only if $B^*\subseteq C^*$; (4) $(\sup\Omega)^*=\cup\{D^*:D\in\Omega\}$ for each directed subset Ω of Spec $\Gamma_L(P)$; (5) $A=\overline{\{a\}}$ if and only if a is an upper bound of A^* and $\bigvee_a^{\uparrow}A^*=a$.

Proof (1) Obviously, A^* is a lower set. Next, we will show that A^* is directed. Let $b,c\in A^*$. We claim that $\Uparrow_L b\cap\Uparrow_L c\cap A\neq\varnothing$. Otherwise, $A=(A-\Uparrow_L b)(A-\Uparrow_L c)$. Then $A=A-\Uparrow_L b$ or $A=A-\Uparrow_L c$ since $A\in$ Spec $\Gamma_L(P)$. Since $b,c\in A^*$, there exist $a_1,a_2\in A$ such that $b\ll_L a_1, c\ll_L a_2$. Thus $a_1\in A-\Uparrow_L b$ or $a_2\in A-\Uparrow_L c$. We get a contradiction. Let $a\in\Uparrow_L b\cap\Uparrow_L c\cap A$. Then there exists a $d\in\Uparrow_L b\cap\Uparrow_L c$ such that $d\ll_L a$ by Lemma 3 and Proposition 1. Thus $d\in A^*$ and d is a common upper bound for b and c.

(2) It is clear that $\overline{A^*}\subseteq A$. For each $a\in A$, we have $\Downarrow_L a\subseteq A^*$ by Definition 2. Then $a=\bigvee_a^{\uparrow}\Downarrow_L a\in\overline{A^*}$ by

Lemma 6, and $A\subseteq\overline{A^*}$. Hence $A=\overline{A^*}$. Therefore, $A=\overline{A^*}=\sup\{\overline{\{b\}}:b\in A^*\}$ by Lemma 7.

(3) Follows directly from (2) and Definition 2.

(4) Let Ω be directed in Spec $\Gamma_L(P)$. Then $\cup\{D^*:D\in\Omega\}$ is directed in P by (1). We claim that $\sup\Omega=\overline{\cup\{D^*:D\in\Omega\}}$. In fact,

$$\begin{aligned}\overline{\cup\{D^*:D\in\Omega\}}&=\sup\{\overline{\{d\}}:d\in\cup\{D^*:D\in\Omega\}\} && \text{by Lemma 7}\\ &=\sup\{\overline{\{d\}}:d\in D^*,D\in\Omega\}=\sup\{\sup\{\overline{\{d\}}:d\in D^*\}:D\in\Omega\}\\ &=\sup\{\overline{D^*}:D\in\Omega\} && \text{by Lemma 7}\\ &=\sup\{D:D\in\Omega\}=\sup\Omega && \text{by (2)}\end{aligned}$$

For each $d\in(\sup\Omega)^*$, there exists a $d^*\in\sup\Omega=\overline{\cup\{D^*:D\in\Omega\}}$ such that $d\ll_L d^*$.

By Lemma 6, there is a bounded directed set $D'\subseteq\cup\{D^*:D\in\Omega\}$ such that $d^*\leqslant\bigvee_b^{\uparrow}D'$, where b is an upper bound of D'. Then there exist $d^{**}\in D', D\in\Omega$ such that $d\leqslant d^{**}$, and $d^{**}\in D^*$. Hence, $d\in D^*$, since D^* is a lower set in P. Thus $(\sup\Omega)^*\subseteq\cup\{D^*:D\in\Omega\}$. On the other hand, by (3), $D^*\subseteq(\sup\Omega)^*$ for each $D\in\Omega$. Then $\cup\{D^*:D\in\Omega\}\subseteq(\sup\Omega)^*$. Therefore, $(\sup\Omega)^*=\cup\{D^*:D\in\Omega\}$.

(5) Assume that $A=\overline{\{a\}}$. Then $\Downarrow_L a\subseteq A^*$ by Definition 2. For each $x\in A^*$, there exists a $y\in A$ such that $x\ll_L y$. By Corollary 1, there exists a bounded directed subset $D\subseteq\downarrow a$ with an upper bound b such that $y\leqslant\bigvee_b^{\uparrow}D$. Hence there exists a $d\in D$ such that $x\leqslant d\leqslant a$. Thus a is an upper bound of A^*. Moreover, $a=\bigvee_a^{\uparrow}\Downarrow_L a\leqslant\bigvee_a^{\uparrow}A^*\leqslant a$. So $a=\bigvee_a^{\uparrow}A^*$. Conversely, let a be an upper bound of A^* and $\bigvee_a^{\uparrow}A^*=a$. So $A=\overline{A^*}=\overline{\{\bigvee_a^{\uparrow}A^*\}}=\overline{\{a\}}$ by (2) and Lemma 7.

Similar to Lemma 8(1), (2), we have the following corollary.

Corollary 2 Let P be an aldcpo. If $A\in$ Spec $\Gamma_L(P)$, then

(1) $A^*\cap K(P)$ is a directed lower set in P; (2) $A=\overline{A^*}=\sup\{\overline{\{b\}}:b\in A^*\cap K(P)\}$.

Lemma 9 Let P be a cldcpo.

(1) If $x\ll_L y$ in P, then $\overline{\{x\}}\ll\overline{\{y\}}$ in Spec $\Gamma_L(P)$; (2) Define $c:P\to$ Spec $\Gamma_L(P)$ by $c(p)=\overline{\{p\}}$ for all $p\in P$. Then the map c is local Scott continuous and preserves the relation $\ll_L$.

Proof (1) Assume that $x\ll_L y$ in P. Let Ω be a directed set in Spec $\Gamma_L(P)$ and $\overline{\{y\}}\subseteq\sup\Omega$. Then $\overline{\{y\}}\subseteq\overline{(\sup\Omega)^*}=\overline{\cup\{D^*:D\in\Omega\}}$ by Lemma 8(2), (4). It follows that $y\in\overline{\cup\{D^*:D\in\Omega\}}$. Hence, by Lemma 6, there is a bounded directed subset $E\subseteq\cup\{D^*:D\in\Omega\}$ such that $y\leqslant\bigvee_b^{\uparrow}E$, where b is an upper bound of E. Thus there exist $e\in E, D\in\Omega$ such that $x\leqslant e$ and $e\in D^*$. Therefore $x\in D^*$, and $\overline{\{x\}}\subseteq\overline{D^*}=D$. Consequently, $\overline{\{x\}}\ll\overline{\{y\}}$.

(2) Obviously, the map c is monotone. Now, it suffices to show that the map c preserves the suprema of bounded directed subset. Suppose that D is a bounded directed set in P and $a=\bigvee_p^{\uparrow}D$, where p is any upper bound of D. Then $c(D)$ is a directed subset of Spec $\Gamma_L(P)$, and $c(d)\subseteq c(a)$ for each $d\in D$. Thus $\sup c(D)=\sup\{c(d):d\in D\}\subseteq c(a)$. Moreover, $D\subseteq\sup c(D)$, since $d\in c(d)\subseteq\sup c(D)$ for each $d\in D$. Then $\overline{D}\subseteq\sup c(D)$, and $c(a)=\overline{D}\subseteq\sup c(D)$ by Lemma 7. Therefore, $c(\bigvee_p^{\uparrow}D)=c(a)=\sup c(D)$. Finally, the map c preserves the relation $\ll_L$ by (1).

Lemma 10 Let P be a cldcpo. Then $X\ll Y$ in Spec $\Gamma_L(P)$ if and only if there are $x,y\in Y$ with $x\ll_L y$ such that $X\subseteq\overline{\{x\}}\subseteq\overline{\{y\}}\subseteq Y$.

Proof Sufficiency. It is clear by Lemma 9(1) and Lemma 4.

Necessity. Assume that $X\ll Y$ in Spec $\Gamma_L(P)$. Since $Y=\sup\{\overline{\{x\}}:x\in Y^*\}$ by Lemma 8(2), there exists an $x\in Y^*$ such that $X\subseteq\overline{\{x\}}$. Moreover, there is a $y\in Y$ such that $x\ll_L y$ by Definition 2. Hence $X\subseteq\overline{\{x\}}\subseteq\overline{\{y\}}\subseteq Y$.

Lemma 11 Let P be a cldcpo. Then $\Downarrow Y=\{\overline{\{y\}}: y\in Y^*\}$ for each $Y\in \mathrm{Spec}\ \Gamma_L(P)$.

Proof Assume that $X\ll Y$ in Spec $\Gamma_L(P)$. Then there exist $y_1, y_2\in Y$ with $y_1\ll_L y_2$ such that $X\subseteq\overline{\{y_1\}}\subseteq\overline{\{y_2\}}\subseteq Y$ by Lemma 10. For each $x\in X^*$, there exists an $x'\in X\subseteq\overline{\{y_1\}}$ such that $x\ll_L x'$ by Definition 2. Then, by Corollary 1, there is a bounded directed set $D\subseteq\downarrow y_1$ such that $x'\leqslant\bigvee_b^{\uparrow}D$, where b is an upper bound of D in P. So, there is a $d\in D$ such that $x\leqslant d$ and $x\leqslant y_1$. Thus y_1 is an upper bound of X^*. Hence $\bigvee_{y_1}^{\uparrow}X^*\in Y^*$, since $\bigvee_{y_1}^{\uparrow}X^*\leqslant y_1\ll_L y_2\in Y$. We have $\overline{\{\bigvee_{y_1}^{\uparrow}X^*\}}\ll\overline{\{y_2\}}\subseteq Y$ by Lemma 4 and Lemma 9(1). In addition, by Lemma 7 and Lemma 8(2), $\overline{\{\bigvee_{y_1}^{\uparrow}X^*\}}=\overline{X^*}=X$. Then $\Downarrow Y\subseteq\{\overline{\{y\}}: y\in Y^*\}$. On the other hand, it follows from Lemma 9(1) that $\{\overline{\{y\}}: y\in Y^*\}\subseteq\Downarrow Y$. Therefore, $\Downarrow Y=\{\overline{\{y\}}: y\in Y^*\}$.

In the following theorem we show that the directed completion of a cldcpo (resp., an aldcpo) is a continuous (resp., algebraic) dcpo.

Theorem 1 Let P be a cldcpo (resp., an aldcpo). Then Spec $\Gamma_L(P)$ is a continuous (resp., an algebraic) dcpo.

Proof For each $Y\in\mathrm{Spec}\ \Gamma_L(P)$, we have $\Downarrow Y=\{\overline{\{y\}}: y\in Y^*\}$ by Lemma 11. Since Y^* is directed in P, $\{\overline{\{y\}}: y\in Y^*\}$ is directed in Spec $\Gamma_L(P)$. Moreover, by Lemma 8(2), $Y=\sup\{\overline{\{y\}}: y\in Y^*\}=\sup\Downarrow Y$. Therefore, Spec $\Gamma_L(P)$ is a continuous dcpo.

Similarly, Spec $\Gamma_L(P)$ is an algebraic dcpo if P is an algebraic local dcpo.

Theorem 2 Let P be a cldcpo. Then for any continuous dcpo Q and lcoal Scott continuous map $f: P\to Q$, there exists a unique Scott continuous map $g:\mathrm{Spec}\ \Gamma_L(P)\to Q$ such that $g\circ c=f$. Moreover, if f preserves the relation $\ll_L$, then g preserves the relation $\ll$.

Proof Define $g:\mathrm{Spec}\ \Gamma_L(P)\to Q$ by $g(X)=\sup\{f(x): x\in X^*\}$ for every $X\in\mathrm{Spec}\ \Gamma_L(P)$.

Claim 1 $y\ll_L x$ if and only if $y\in\overline{\{x\}}^*$.

Sufficiency. Assume that $y\in\overline{\{x\}}^*$. Then there exists a $y'\in\overline{\{x\}}$ such that $y\ll_L y'$ by Definition 2. Hence, by Corollary 1, there exists a bounded directed set $D\subseteq\downarrow x$ such that $y'\leqslant\bigvee_b^{\uparrow}D$, where b is an upper bound of D. In addition, there is a $z\in P$ such that $y\ll_L z\ll_L y'$ by Lemma 2. So, there exists a $d\in D$ such that $z\leqslant d$. Thus $y\ll_L z\leqslant d\leqslant x$. Therefore $y\ll_L x$ by Lemma 4.

Necessity. Obviously, $y\in\overline{\{x\}}^*$, since $x\in\overline{\{x\}}$.

Claim 2 $g\circ c=f$.

For each $x\in P$, we have

$$\begin{aligned}
g(\overline{\{x\}}) &= \bigvee^{\uparrow}\{f(y): y\in\overline{\{x\}}^*\} && \text{by the definition of } g\\
&= \bigvee^{\uparrow}\{f(y): y\ll_L x\} && \text{by Claim 1}\\
&= f(\textstyle\bigvee_x^{\uparrow}\Downarrow x) && \text{by } f \text{ being a local Scott continuous map}\\
&= f(x) && \text{by } P \text{ being a continuous local dcpo.}
\end{aligned}$$

Hence $g\circ c=f$.

Claim 3 The map g preserves suprema of directed sets.

Obviously, g is order preserving by the definition of g. Assume that Ω is any directed set of Spec $\Gamma_L(P)$. Next, it suffices to show $g(\sup\Omega)=\sup\{g(D): D\in\Omega\}$. In fact,

$$\begin{aligned}
g(\sup\Omega) &= \sup\{f(d): d\in(\sup\Omega)^*\} && \text{by the definition of } g\\
&= \sup\{f(d): d\in\cup\{D^*: D\in\Omega\}\} && \text{by Lemma 8(4)}\\
&= \sup\{f(d): d\in D^*, D\in\Omega\}=\sup\{\sup\{f(d): d\in D^*\}: D\in\Omega\}\\
&= \sup\{g(D): D\in\Omega\} && \text{by the definition of } g.
\end{aligned}$$

Claim 4 The map g preserves the relation $\ll$ if f preserves the relation $\ll_L$.

Assume that $X \ll Y$. Then there exist $y_1, y_2 \in Y$ such that $y_1 \ll_L y_2$ and $X \subseteq \overline{\{y_1\}} \subseteq \overline{\{y_2\}} \subseteq Y$ by Lemma 10. Thus $f(y_1) \ll f(y_2)$, since f preserves the relation $\ll_L$. Moreover, $g(X) \subseteq g(\overline{\{y_1\}}) = g \circ c(y_1) = f(y_1) \ll f(y_2) = g \circ c(y_2) = g(\overline{\{y_2\}}) \subseteq g(Y)$. Hence $g(X) \ll g(Y)$.

Claim 5 $h = g$ if there exists a Scott continuous map h satisfying $h \circ c = f$. For each $X \in \mathrm{Spec}\ \Gamma_L(P)$,

$$\begin{aligned} h(X) &= h(\sup\{\overline{\{x\}} : x \in X^*\}) && \text{by Lemma 8(2)} \\ &= \sup\{h(\overline{\{x\}}) : x \in X^*\} && \text{by } h \text{ preserving suprema of directed sets} \\ &= \sup\{f(x) : x \in X^*\} && \text{by } h \circ c = f \\ &= g(X) && \text{by the definition of } g. \end{aligned}$$

The proof is finished.

Let CLDcpo(resp., ALDcpo) be the category of continuous(resp., algebraic) local dcpos and local Scott continuous maps preserving $\ll_L$; Let CDcpo(resp., ADcpo) be the category of continuous(resp., algebraic) dcpos and Scott continuous maps preserving $\ll$.

By Proposition 3.3.6[5] and Theorem 2, we have the following theorem.

Theorem 3 CDcpo(resp., ADcpo) is a full reflective subcategory of CLDcpo(resp., ALDcpo).

[Reference]

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