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非等熵 Euler-Maxwell 方程组大稳态解的稳定性

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[**摘要**]　研究无温度衰减的非等熵 Euler-Maxwell 方程组在非常数大稳态解附近周期光滑解的稳定性. 通过引入新的变量及一个非对角的对称化子,并借助反对称矩阵的性质和归纳法,给出了稳定性结果的简洁证明.

[**关键词**]　非等熵 Euler-Maxwell 方程组,整体光滑解;稳定性,能量估计

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Stability of Large Steady-State Solutions to Non-Isentropic Euler-Maxwell Systems

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Abstract:Stability of periodic smooth solutions near non-constant steady-states for a non-isentropic Euler-Maxwell system without temperature damping term are studied. New variables are introduced and choose a non-diagonal symmetrizer of the full Euler equations to recover dissipation estimates. The proof is based on an induction argument on the order of the derivatives of solutions in energy and time dissipation estimates. This allows to make the proof of the stability result very simple and concise.

Key words:non-isentropic Euler-Maxwell system;global smooth solutions;stability;energy estimates

本文主要研究被磁化的等离子体中非等熵 Euler-Maxwell 方程组在三维环面 $T^3=(R/Z)^3$ 上周期光滑解在大稳态解附近的稳定性(见[1-4]):

$$\begin{cases}\partial_t n+\mathrm{div}(n\boldsymbol{u})=0,\\ \partial_t(n\boldsymbol{u})+\mathrm{div}(n\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p(n)=-n(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})-n\boldsymbol{u},\\ \partial_t\theta+\boldsymbol{u}\cdot\nabla\theta+(\gamma-1)\theta\mathrm{div}\boldsymbol{u}=\dfrac{\gamma-1}{2R}|\boldsymbol{u}|^2-(\theta-\theta_L),\\ \partial_t\boldsymbol{E}-\nabla\times\boldsymbol{B}=n\boldsymbol{u},\quad \mathrm{div}\boldsymbol{E}=b-n,\\ \partial_t\boldsymbol{E}+\nabla\times\boldsymbol{E}=0,\quad \mathrm{div}\boldsymbol{B}=0.\end{cases}\tag{1}$$

式中,$n,\boldsymbol{u}\in R^3,\theta$ 分别代表电子的密度,速度和温度,$\boldsymbol{E}$ 和 $\boldsymbol{B}$ 分别代表 3 维空间的电场和磁场,以上变量都是位置 $\boldsymbol{x}\in R^3$ 和时间 $t>0$ 的未知函数. 电场 $\boldsymbol{E}$ 和磁场 $\boldsymbol{B}$ 通过 Maxwell 方程和$(n,\boldsymbol{u},\theta)$耦合在一起并通过洛伦兹力 $n(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})$作用在粒子上. 这里 b 表示离子的密度,为了保证 b 和方程组(1)是相容的,则 b 仅依赖于空间变量 $\boldsymbol{x}$ 且 $b=b(\boldsymbol{x})$是光滑周期函数. 此外,假设在 $\boldsymbol{x}\in T^3$ 上 $b\geqslant const.>0$.

事实上(1)描述的的对象是理想多方气体,即,状态方程为

$$p=Rn\theta,e=C_v\theta.\tag{2}$$

式中,$R>0,C_v>0$ 且 R 和 C_v 都是常数. 由等式(2),记

$$\gamma=\frac{R+C_v}{C_v}>1,\theta_L=\frac{e_L}{C_v}.$$

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本文主要研究系统(1)赋如下初始条件时其周期问题光滑解的整体存在性和渐近性.

$$t=0:(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})=(n_0,\boldsymbol{u}_0,\theta_0,\boldsymbol{E}_0,\boldsymbol{B}_0),\quad \boldsymbol{x}\in T^3. \tag{3}$$

对于任意非真空场中的光滑解,可以将(1)中的动量方程写作

$$\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{1}{n}\nabla p=-(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B})-\boldsymbol{u}. \tag{4}$$

(4)右边的项$-\boldsymbol{u}$和(1)的能量方程$-(\theta-\theta_L)$是能量估计中的速度衰减项和温度衰减项.

由(1)的约束方程,可设

$$\mathrm{div}\boldsymbol{E}_0=b-n_0,\mathrm{div}\boldsymbol{B}_0=0. \tag{5}$$

现在考虑当$\bar{\boldsymbol{u}}=0$时,(1)在稳定状态下的解,令$(\bar{n},\bar{\boldsymbol{u}},\bar{\boldsymbol{E}},\bar{\boldsymbol{B}})$是仅依赖于变量$\boldsymbol{x}$的一个解且$\bar{\boldsymbol{u}}=0$,则有

$$\begin{cases}\nabla h(\bar{n})=-\bar{\boldsymbol{E}},\\ \nabla\times\bar{\boldsymbol{B}}=0,\quad \mathrm{div}\bar{\boldsymbol{E}}=b-\bar{n},\\ \nabla\times\bar{\boldsymbol{E}}=0,\quad \mathrm{div}\bar{\boldsymbol{B}}=0.\end{cases} \tag{6}$$

故$\bar{\boldsymbol{B}}$是常数向量,$\bar{n}$在T^3中满足椭圆方程

$$-\Delta h(\bar{n})=b-\bar{n}. \tag{7}$$

当$\bar{n}$解出,由(7)可知$\bar{\boldsymbol{E}}=-\nabla h(\bar{n})$,所以$(\bar{n},\bar{\boldsymbol{u}},\bar{\boldsymbol{E}})$是稳定状态下的一个特殊解且$\bar{\boldsymbol{u}}=0$. 令$\bar{\phi}=h(\bar{n})$,$h^{-1}$是$h$的反函数,则(7)在$T^3$中等价于一个半线性椭圆方程:$-\Delta\bar{\phi}=b-h^{-1}(\bar{\phi})$. 由 Schauder 不动点定理或极小值方法容易得到这个方程光滑解的存在性,由h的严格单调性可得光滑解的唯一性. 根据 Neumann 边界条件[5]或 Dirichlet 条件[6],可以在有解范围内得到光滑解的存在唯一性. 由极大值原理且$b\geqslant const.>0$可知这些解满足$\bar{n}\geqslant const.>0$. 所以在周期情况下,(7)周期光滑解的存在唯一性是明显的.

命题 1(见[7-8])　假设b是T^3中的一个光滑周期函数且$b\geqslant const.>0$,则周期问题(7)存在唯一光滑解$\bar{n}\in T^3$且$\bar{n}\geqslant const.>0$.

当$n,\theta>0$时,Euler-Maxwell 系统(1)是一个对称双曲型方程组(见[7]),若初值是光滑的,则(1)-(2)存在唯一的局部光滑解(见[8])(关于一维的情形见[9-10]).

命题 2(见[8,11])　令s是一个整数且$s\geqslant3$,$\bar{\boldsymbol{B}}$是一个给定的常数向量且$\bar{\boldsymbol{B}}\in R^3$,$\bar{n}$是由命题 1 给出(7)的一个光滑解且$\bar{\boldsymbol{E}}=-\nabla h(\bar{n})$. 假设(5)成立且当$n_0\geqslant2\kappa$,$\kappa$是常数($\kappa>0$)时有$(n_0-\bar{n},\boldsymbol{u}_0,\theta-\theta_L,\boldsymbol{E}_0-\bar{\boldsymbol{E}},\boldsymbol{B}_0-\bar{\boldsymbol{B}})\in H^s(T^3)$. 则存在$T_1>0$使得(1)-(3)在$[0,T_1]\times T^3$中存在唯一光滑解$(n,\boldsymbol{u},\theta,\boldsymbol{E},\boldsymbol{B})$满足$n\geqslant\kappa$且$(n-\bar{n},\boldsymbol{u},\theta-\theta_L,\boldsymbol{E}-\bar{\boldsymbol{E}},\boldsymbol{B}-\bar{\boldsymbol{B}})\in C^1([0,T_1];H^{s-1}(T^3))\cap C([0,T_1];H^s(T^3))$.

在一维非等熵的 Euler-Maxwell 系统中,通过补偿紧方法可获得熵解的整体存在性(见[12-13]). 关于平衡状态下整体光滑解的存在性问题已经有很多研究. 在[4]中,彭跃军主要研究在磁化的等离子体模型中可压缩的等熵 Euler-Maxwell 方程的周期问题. 该方程是部分耗散的拟线性双曲方程组. 证明了当时间趋于无穷时,存在整体光滑解且在非常数平衡态下收敛. 证明主要是通过对解的各阶导数进行归纳来实现,并证明了可压缩的 Euler-Poisson 方程的解指数衰减到稳定状态. 在[14]中,王术等人主要研究带有额外温度耗散项的非等熵 Euler-Maxwell 系统在平衡状态下的周期问题,证明在非常数平衡态下光滑解的全局存在性及当时间趋于无穷时解收敛到稳定状态. 最近,刘存明与彭跃军在[2]中主要研究没有温度耗散项的非等熵 Euler-Poisson 方程在非常数平衡态下周期光滑解的稳定性,证明了当时间趋于无穷大时,解以指数方式衰减到稳定状态. 主要通过引入新的压强变量和选择 Euler 方程的一个非对角的对称化子获得耗散估计. 此外,在[15]中,王术等人主要研究可压非等熵 Euler-Maxwell 系统在常数平衡态下解的全局存在性和渐近衰减.

1　预备知识

首先介绍一些记号. 当$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\in N^3$时,记$\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}=\dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial_{x_1}^{\alpha_1}\partial_{x_1}^{\alpha_2}\partial_{x_1}^{\alpha_3}}$,且$|\boldsymbol{\alpha}|=\alpha_1+\alpha_2+\alpha_3$. 其中$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$,$\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)$,$\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}$代表的是$\boldsymbol{\beta}_j\leqslant\boldsymbol{\alpha}_j(j=1,2,3)$,记 Sobolev 空间$H^s(T^3)$的范数为$\|\cdot\|_s$,$L^2(T^3)$和$L^\infty(T^3)$空间的范数分别记为$\|\cdot\|$和$\|\cdot\|_\infty$.

Leibniz 公式:$\partial_x^{\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{v})=\sum_{\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}} m_{\boldsymbol{\alpha\beta}}\partial_x^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\boldsymbol{u}\partial_x^{\boldsymbol{\beta}}\boldsymbol{v}$, $\forall\boldsymbol{\alpha}\in N^3$,其中 $m_{\boldsymbol{\alpha\beta}}$ 是常数.

$$\partial_t^k(\boldsymbol{u}\boldsymbol{v})=\sum_{l=0}^{k} m'_{kl}\partial_t^{k-l}\boldsymbol{u}\partial_t^l\boldsymbol{v},\ \forall k\in N,\text{其中 } m'_{kl}\text{是常数.}$$

$B_{s,T}(T^3)$ 是一个 Banach 空间定义为:

$$B_{s,T}(T^3)=\bigcap_{k=0}^{s}C^k([0,T];H^{s-k}(T^3)),$$

它的范数记为:

$$\|\boldsymbol{v}\|_{B_{s,T}}=\max_{0\leqslant t\leqslant T}|||\boldsymbol{v}(t,\cdot)|||_s,\ \forall\boldsymbol{v}\in B_{s,T}(T^3),$$

式中,$|||\boldsymbol{v}(t,\cdot)|||_s=\left(\sum_{|\boldsymbol{\alpha}|+k\leqslant s}\|\partial_t^k\partial_x^{\boldsymbol{\alpha}}\boldsymbol{v}(t,\cdot)\|^2\right)^{1/2}$.

引理 1　假设 $\boldsymbol{u},\boldsymbol{v}\in H^s(T^3)$,$f$ 是一个光滑函数,若 $\boldsymbol{\alpha}\in N^3$ 且 $1\leqslant|\boldsymbol{\alpha}|\leqslant s$,记 $\|D^{s'}\boldsymbol{u}\|=\sum_{|\boldsymbol{\alpha}|=s'}\|\partial_x^{\boldsymbol{\alpha}}\boldsymbol{u}\|$,则有下面的不等式成立

$$\|\partial_x^{\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{v})-\boldsymbol{u}\partial_x^{\boldsymbol{\alpha}}\boldsymbol{v}\|\leqslant C_s(\|\nabla\boldsymbol{u}\|_\infty\|D^{|\boldsymbol{\alpha}|-1}\boldsymbol{v}\|+\|D^{|\boldsymbol{\alpha}|}\boldsymbol{u}\|\|\boldsymbol{v}\|_\infty).\tag{8}$$

$$\|\partial_x^{\boldsymbol{\alpha}}f(\boldsymbol{u})\|\leqslant C_\infty(1+\|\nabla\boldsymbol{u}\|_\infty)^{|\boldsymbol{\alpha}|-1}\|D^{|\boldsymbol{\alpha}|}\boldsymbol{u}\|.\tag{9}$$

式中,$C_s>0$ 是一个仅依赖于 s 的常数,$C_\infty>0$ 是一个依赖于 f,$\|\boldsymbol{u}\|_\infty$ 和 s 的常数. 若 $s>\frac{3}{2}+1$,则

$$\|\partial_x^{\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{v})-\boldsymbol{u}\partial_x^{\boldsymbol{\alpha}}\boldsymbol{v}\|\leqslant C_s\|\nabla\boldsymbol{u}\|_{s-1}\|\boldsymbol{v}\|_{|\boldsymbol{\alpha}|-1}.\tag{10}$$

$$\|\partial_x^{\boldsymbol{\alpha}}(\boldsymbol{u}\boldsymbol{v})\|\leqslant C_s\|\nabla\boldsymbol{u}\|_s\|\boldsymbol{v}\|_{|\boldsymbol{\alpha}|}(|\boldsymbol{\alpha}|\leqslant s).\tag{11}$$

不等式(8)-(9)证明见[11,16],(10)-(11)证明见[4].

引理 2 如果 f 是一个光滑函数,$\boldsymbol{u},\boldsymbol{v}\in B_{s,T}(T^3)$ 且 $s>\frac{3}{2}+1$. $k,l\in N$ 且 $\boldsymbol{\alpha},\boldsymbol{\beta}\in N^3$. 见[15].

(1)若 $k+|\boldsymbol{\alpha}|\leqslant s$,则

$$\|\partial_x^{\boldsymbol{\alpha}}(\boldsymbol{u}\partial_t^k\nabla\boldsymbol{v})-\boldsymbol{u}\partial_x^{\boldsymbol{\alpha}}\partial_t^k\nabla\boldsymbol{v}\|\leqslant C_s|||\boldsymbol{u}|||_s|||\boldsymbol{v}|||_s.\tag{12}$$

(2)若 $k+|\boldsymbol{\alpha}|\leqslant s$,$l+|\boldsymbol{\beta}|\leqslant s$,$k+l\leqslant s$,$|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|\leqslant s$,$k+l+|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|\leqslant s+1$. 则

$$\|(\partial_t^k\partial_x^{\boldsymbol{\alpha}}\boldsymbol{u})(\partial_t^l\partial_x^{\boldsymbol{\beta}}\boldsymbol{v})\|\leqslant C_s|||\boldsymbol{u}|||_s|||\boldsymbol{v}|||_s.\tag{13}$$

此外若 $l+|\boldsymbol{\alpha}|\geqslant 1$,$k+|\boldsymbol{\beta}|\geqslant 1$ 则

$$\|(\partial_t^k\partial_x^{\boldsymbol{\alpha}}\boldsymbol{u})(\partial_t^l\partial_x^{\boldsymbol{\beta}}\boldsymbol{v})\|\leqslant C_s|||\partial\boldsymbol{u}|||_{s-1}|||\partial\boldsymbol{v}|||_{s-1}.\tag{14}$$

式中,∂ 代表对 t 和 x 任意阶导数,所以

$$|||\boldsymbol{u}\boldsymbol{v}|||_s\leqslant C_s|||\boldsymbol{u}|||_s|||\boldsymbol{v}|||_s.\tag{15}$$

(3)若 $1\leqslant k$,$k+|\boldsymbol{\alpha}|\leqslant s$,则

$$\|\partial_t^k\partial_x^{\boldsymbol{\alpha}}f(\boldsymbol{v})\|\leqslant C_s|||\partial_t\boldsymbol{v}|||_s.\tag{16}$$

2　方程组的对称化

令

$$N=n-\bar{n},\ \Theta=\theta-\theta_L,\ U=\begin{pmatrix}N\\ \boldsymbol{u}\\ \Theta\end{pmatrix},\ \boldsymbol{B}=\bar{\boldsymbol{B}}+\boldsymbol{G},\ \boldsymbol{F}=\boldsymbol{E}-\bar{\boldsymbol{E}}=\boldsymbol{E}+\nabla h(\bar{n}).\tag{17}$$

当 $k\in N$,$\boldsymbol{\alpha}\in N^3$ 且 $k+|\boldsymbol{\alpha}|\leqslant s$ 时,记 $\boldsymbol{U}_{k,\boldsymbol{\alpha}}=\partial_t^k\partial_x^{\boldsymbol{\alpha}}\boldsymbol{U}$. 用 $\ln p$ 代替 n 证明稳定性结果,容易发现压强 p 满足

$$\partial_t p+\boldsymbol{u}\cdot\nabla p+\gamma p\,\mathrm{div}\boldsymbol{u}=\frac{(\gamma-1)p}{2R\theta}|\boldsymbol{u}|^2-\frac{p}{\theta}\Theta.\tag{18}$$

令 $q=\ln p$,$\bar{q}=\ln\bar{p}$. 当 $p>0$ 时,得:

$$\partial_t q+\boldsymbol{u}\cdot\nabla q+\gamma\,\mathrm{div}\boldsymbol{u}=\frac{\gamma-1}{2R\theta}|\boldsymbol{u}|^2-\frac{1}{\theta}\Theta.\tag{19}$$

令

$$Q=q-\bar{q},\boldsymbol{W}=\begin{pmatrix}\boldsymbol{W}^1\\ \boldsymbol{W}^2\end{pmatrix},\boldsymbol{W}^1=\begin{pmatrix}Q\\ \boldsymbol{u}\\ \boldsymbol{\Theta}\end{pmatrix},\boldsymbol{W}^2=\begin{pmatrix}\boldsymbol{F}\\ \boldsymbol{G}\end{pmatrix}. \tag{20}$$

将这些变量代入方程(1)和(19)，得

$$\begin{cases}\partial_t Q+\boldsymbol{u}\cdot\nabla Q+\gamma\mathrm{div}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\bar{q}=\dfrac{\gamma-1}{2R\theta}|\boldsymbol{u}|^2-\dfrac{1}{\theta}\boldsymbol{\Theta},\\ \partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+R\theta\nabla Q+R\nabla\bar{q}\boldsymbol{\Theta}=-(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))-\boldsymbol{u},\\ \partial_t\boldsymbol{\Theta}+\boldsymbol{u}\cdot\nabla\boldsymbol{\Theta}+(\gamma-1)\theta\mathrm{div}\boldsymbol{u}=\dfrac{\gamma-1}{2R}|\boldsymbol{u}|^2-\boldsymbol{\Theta},\\ \partial_t\boldsymbol{F}-\nabla\times\boldsymbol{G}=(\bar{n}+N)\boldsymbol{u},\quad \mathrm{div}\boldsymbol{F}=-N,\\ \partial_t\boldsymbol{G}+\nabla\times\boldsymbol{F}=0,\quad \mathrm{div}\boldsymbol{G}=0.\end{cases} \tag{21}$$

(21)可写成如下形式

$$\partial_t\boldsymbol{W}^1+\sum_{j=1}^{d}\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{W}^1+\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}^1=\boldsymbol{K}(\boldsymbol{u},\theta,\boldsymbol{F},\boldsymbol{G}). \tag{22}$$

式中，$\boldsymbol{A}_j(\boldsymbol{u},\theta)=\begin{pmatrix}u_j & \gamma\boldsymbol{e}_j^T & 0\\ R\theta\boldsymbol{e}_j & u_j\boldsymbol{I}_3 & 0\\ 0 & (\gamma-1)\theta\boldsymbol{e}_j^T & u_j\end{pmatrix}$，$j=1,2,3.$

$$\boldsymbol{L}(\boldsymbol{x})=\begin{pmatrix}0 & (\nabla\bar{q})^T & 0\\ 0 & 0 & R\nabla\bar{q}\\ 0 & 0 & 0\end{pmatrix},\boldsymbol{K}(\boldsymbol{u},\theta,\boldsymbol{F},\boldsymbol{G})=\begin{pmatrix}\dfrac{\gamma-1}{2R\theta}|\boldsymbol{u}|^2-\dfrac{1}{\theta}(\theta-\theta_L)\\ -(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))-\boldsymbol{u}\\ \dfrac{\gamma-1}{2R}|\boldsymbol{u}|^2-(\theta-\theta_L)\end{pmatrix}.$$

(22)的初始条件是

$$t=0:\boldsymbol{W}=\boldsymbol{W}_0\overset{\text{def}}{=\!=}(Q_0,\boldsymbol{u}_0,\boldsymbol{\Theta}_0,\boldsymbol{E}_0,\boldsymbol{B}_0),\quad \boldsymbol{x}\in T^3. \tag{23}$$

式中，$Q_0=\ln(Rn_0\theta_0)-\ln(R\bar{n}\theta_L)$，$\boldsymbol{\Theta}_0=\theta_0-\theta_L$. 容易得到一个非对角矩阵(见[15])

$$\boldsymbol{A}_0(p,\theta)=\begin{pmatrix}Rp & 0 & -\dfrac{Rp}{\theta}\\ 0 & \dfrac{p}{\theta}\boldsymbol{I}_3 & 0\\ -\dfrac{Rp}{\theta} & 0 & \dfrac{R\gamma p}{(\gamma-1)\theta^2}\end{pmatrix}. \tag{24}$$

当 $p>0,\theta>0$ 时，(24)是对称且正定的矩阵.

$$\tilde{\boldsymbol{A}}_j(p,\boldsymbol{u},\theta)\overset{\text{def}}{=\!=}\boldsymbol{A}_0(p,\theta)\boldsymbol{A}_j(\boldsymbol{u},\theta)=\begin{pmatrix}Rpu_j & Rp\boldsymbol{e}_j^T & -\dfrac{Rp}{\theta}u_j\\ Rp\boldsymbol{e}_j & \dfrac{p}{\theta}u_j\boldsymbol{I}_3 & 0\\ -\dfrac{Rp}{\theta}u_j & 0 & \dfrac{R\gamma p}{(\gamma-1)\theta^2}u_j\end{pmatrix}.$$

是对称矩阵，故方程(21)是对称双曲的. 此外，

$$\boldsymbol{A}_0(p,\theta)\boldsymbol{L}(\boldsymbol{x})=\begin{pmatrix}0 & Rp(\nabla\bar{q})^T & 0\\ 0 & 0 & \dfrac{Rp}{\theta}\nabla\bar{q}\\ 0 & -\dfrac{Rp}{\theta}(\nabla\bar{q})^T & 0\end{pmatrix}.$$

令

$$\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})=\sum_{j=1}^{3}\partial_{x_j}\widetilde{\boldsymbol{A}}_j(p,\boldsymbol{u},\theta)-2\boldsymbol{A}_0(p,\theta)\boldsymbol{L}(\boldsymbol{x}).$$

由 $\nabla\bar{q}=\dfrac{1}{\bar{p}}\nabla\bar{p}$,可得:

$$\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})=\begin{pmatrix} R\mathrm{div}(p\boldsymbol{u}) & R(\nabla p)^T-\dfrac{2Rp}{\bar{p}}(\nabla\bar{p})^T & -R\mathrm{div}(\dfrac{p\boldsymbol{u}}{\theta}) \\ R\nabla p & \mathrm{div}(\dfrac{p\boldsymbol{u}}{\theta})\boldsymbol{I}_3 & -\dfrac{2Rp}{\theta\bar{p}}\nabla\bar{p} \\ -R\mathrm{div}(\dfrac{p\boldsymbol{u}}{\theta}) & \dfrac{2Rp}{\theta\bar{p}}\nabla\bar{p} & \dfrac{R\gamma}{(\gamma-1)}\mathrm{div}(\dfrac{p\boldsymbol{u}}{\theta^2}) \end{pmatrix}.$$

令 $T>0$,$\boldsymbol{W}$ 是(22)-(23)定义在$[0,T]$上的一个光滑解,记

$$\boldsymbol{W}_T=\sup_{t\in[0,T]}|||\boldsymbol{W}(t,\cdot)|||_s.$$

本文假设 $s>\dfrac{d}{2}+1$ 且 $d\leqslant 3$,$\boldsymbol{W}_T$ 是足够小的,且满足

$$\frac{\bar{n}}{2}\leqslant n\leqslant\frac{3\bar{n}}{2},\frac{\bar{p}}{2}\leqslant p\leqslant\frac{3\bar{p}}{2},|\boldsymbol{u}|\leqslant\frac{1}{2},\frac{\theta_L}{2}\leqslant\theta\leqslant\frac{3\theta_L}{2}. \tag{25}$$

3　能量估计

当 $k\in N$,$\boldsymbol{\alpha}\in N^3$ 且 $k+|\boldsymbol{\alpha}|\leqslant s$ 时,用$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}$ 作用(22),可得:

$$\partial_t\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1+\sum_{j=1}^{d}\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1+\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1=\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K}+\boldsymbol{g}^{k,\boldsymbol{\alpha}}. \tag{26}$$

$$\boldsymbol{g}^{k,\boldsymbol{\alpha}}=\sum_{j=1}^{3}\left[\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{W}^1)\right]+\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}^1) \tag{27}$$

对于 Maxwell 方程,有

$$\begin{cases}\partial_t(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{F})-\nabla\times(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{G})=\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(n\boldsymbol{u}), & \mathrm{div}(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{F})=-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{N}. \\ \partial_t(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{G})+\nabla\times(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{F})=0, & \mathrm{div}(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{G})=0.\end{cases} \tag{28}$$

引理 3　当 $k\in N$,$\boldsymbol{\alpha}\in N^3$ 且 $1\leqslant|\boldsymbol{\alpha}|$,$k+|\boldsymbol{\alpha}|\leqslant s$ 时,有

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle\boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))+C_0(\|\boldsymbol{u}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{\Theta}_{k,\boldsymbol{\alpha}}\|^2)\leqslant$$
$$C(\|\partial_t^k(\boldsymbol{F},\boldsymbol{u},\boldsymbol{\Theta})\|_{|\boldsymbol{\alpha}|-1}^2+\|\partial_t^kQ\|_{|\boldsymbol{\alpha}|}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{29}$$

证明　$\boldsymbol{A}_0(p,\theta)\boldsymbol{W}_{k,\boldsymbol{\alpha}}$与(26)作内积得

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{A}_0(p,\theta)\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle=\langle\partial_t\boldsymbol{A}_0(p,\theta)\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+\langle\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+$$
$$2\langle A_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+2\langle\boldsymbol{A}_0\boldsymbol{g}^{k,\boldsymbol{\alpha}},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle=I_1+I_2+I_3+I_4.$$

I_1 的估计:由(1)中的能量方程和(19)以及 Sobolev 嵌入定理可得:

$$\|(\partial_tp,\partial_t\theta)\|_\infty\leqslant C\|(\boldsymbol{u},\nabla\boldsymbol{u},\boldsymbol{\Theta})\|_\infty\leqslant C|||\boldsymbol{W}|||_s.$$

所以:

$$|\langle\partial_t\boldsymbol{A}_0(p,\theta)\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle|\leqslant C\|(\partial_tp,\partial_t\theta)\|_\infty\|\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\|^2\leqslant C|||\boldsymbol{W}|||_s^3. \tag{30}$$

I_2 的估计:当$(p,\boldsymbol{u},\theta)=(\bar{p},0,\theta_L)$时,矩阵 $\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})$是反对称矩阵,得到:

$$|\langle\boldsymbol{B}(p,\boldsymbol{u},\theta,\boldsymbol{x})\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle|\leqslant C|||\boldsymbol{W}|||_s^3. \tag{31}$$

I_3 的估计:可以将 I_3 写成如下形式:

$$2\langle\boldsymbol{A}_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle=2\langle\boldsymbol{A}_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K}_1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+2\langle\boldsymbol{A}_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K}_2,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle. \tag{32}$$

式中,

$$\boldsymbol{K}_1(\boldsymbol{u},\theta,\boldsymbol{F},\boldsymbol{G})=\begin{pmatrix}-\frac{1}{\theta}\Theta\\-(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))-\boldsymbol{u}\\-\Theta\end{pmatrix},\quad \boldsymbol{K}_2(\boldsymbol{u},\theta)=\begin{pmatrix}\frac{\gamma-1}{2R\theta}|\boldsymbol{u}|^2\\0\\\frac{\gamma-1}{2R}|\boldsymbol{u}|^2\end{pmatrix}.\tag{33}$$

首先考虑(32)右边的第二项有：

$$2|\langle \boldsymbol{A}_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K}_2,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle|\leqslant C|||\boldsymbol{W}|||_s^3.\tag{34}$$

再考虑(32)右边的第一项得：

$$2\langle \boldsymbol{A}_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K}_1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle=-2R\langle n\boldsymbol{u}_{k,\boldsymbol{\alpha}},\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle-2R\langle(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))_{k,\boldsymbol{\alpha}},n\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle+$$
$$\langle 2Rp(\frac{\Theta_{k,\boldsymbol{\alpha}}}{\theta}-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\frac{\Theta}{\theta})),Q_{k,\boldsymbol{\alpha}}\rangle+\langle\frac{2Rp}{\theta}\left(\left(\frac{\Theta}{\theta}\right)_{k,\boldsymbol{\alpha}}-\frac{\gamma}{(\gamma-1)\theta}\Theta_{k,\boldsymbol{\alpha}}\right),\Theta_{k,\boldsymbol{\alpha}}\rangle.\tag{35}$$

由(28)得：

$$-R\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2)=-2R\langle\partial_t^k\partial_x^{\alpha}(n\boldsymbol{u}),\boldsymbol{F}_{k,\boldsymbol{\alpha}}\rangle.$$

(35)右边的第二项估计：

$$-2R\langle(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))_{k,\boldsymbol{\alpha}},n\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle=-2R\langle\boldsymbol{F}_{k,\boldsymbol{\alpha}},n\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle+2R\langle(\nabla h(\bar{n})-\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))_{k,\boldsymbol{\alpha}},n\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle=$$
$$-R\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2)+2R\langle(\nabla h(\bar{n})-\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))_{k,\boldsymbol{\alpha}},(N+\bar{n})\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle+2R\langle\boldsymbol{F}_{k,\boldsymbol{\alpha}},(n\boldsymbol{u})_{k,\boldsymbol{\alpha}}-n\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle\leqslant$$
$$-R\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2)+R\langle n\boldsymbol{u}_{k,\boldsymbol{\alpha}},\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle+C\|\partial_t^k\boldsymbol{u}\|_{|\boldsymbol{\alpha}|-1}^2+C\|\partial_t^k\boldsymbol{F}\|_{|\boldsymbol{\alpha}|-1}^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s+\varepsilon\|\boldsymbol{u}_{k,\boldsymbol{\alpha}}\|^2.\tag{36}$$

(35)右边的第三项估计：

$$\left|\langle 2Rp\left(\frac{\Theta_{k,\boldsymbol{\alpha}}}{\theta}-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\left(\frac{\Theta}{\theta}\right)\right),Q_{k,\boldsymbol{\alpha}}\rangle\right|\leqslant C|||\Theta|||_s^2|||Q|||_s.\tag{37}$$

(35)右边的第四项估计：

$$\langle\frac{2Rp}{\theta}\left(\left(\frac{\Theta}{\theta}\right)_{k,\boldsymbol{\alpha}}-\frac{\gamma}{(\gamma-1)\theta}\Theta_{k,\boldsymbol{\alpha}}\right),\Theta_{k,\boldsymbol{\alpha}}\rangle\leqslant-\langle\frac{2Rp}{(\gamma-1)\theta^2}\Theta_{k,\boldsymbol{\alpha}},\Theta_{k,\boldsymbol{\alpha}}\rangle+C|||\Theta|||_s^3.\tag{38}$$

由(34)-(38)得：

$$2\langle\boldsymbol{A}_0\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}\boldsymbol{K},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2)+R\langle n\boldsymbol{u}_{k,\boldsymbol{\alpha}},\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle+\langle\frac{2Rp}{(\gamma-1)\theta^2}\Theta_{k,\boldsymbol{\alpha}},\Theta_{k,\boldsymbol{\alpha}}\rangle\leqslant$$
$$C\|\partial_t^k\boldsymbol{u}\|_{|\boldsymbol{\alpha}|-1}^2+C\|\partial_t^k\boldsymbol{F}\|_{|\boldsymbol{\alpha}|-1}^2+\varepsilon\|\boldsymbol{u}_{k,\boldsymbol{\alpha}}\|^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s^4.\tag{39}$$

I_4 的估计：由(29)，可令 $\boldsymbol{g}^{k,\boldsymbol{\alpha}}=\boldsymbol{g}_1^{k,\boldsymbol{\alpha}}+\boldsymbol{g}_2^{k,\boldsymbol{\alpha}}$. (40)

式中，

$$\boldsymbol{g}_1^{k,\boldsymbol{\alpha}}=\sum_{j=1}^{3}[\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{A}_j(\boldsymbol{u},\theta)\partial_{x_j}\boldsymbol{W}^1].\tag{41}$$

$$\boldsymbol{g}_2^{k,\boldsymbol{\alpha}}=\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}^1).\tag{42}$$

由引理 1-2，有：

$$2|\langle\boldsymbol{A}_0\boldsymbol{g}_1^{k,\boldsymbol{\alpha}},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle|\leqslant C|||\boldsymbol{W}|||_s^3.\tag{43}$$

由 Leibniz 公式可得：

$$\boldsymbol{g}_2^{k,\boldsymbol{\alpha}}=-\sum_{\boldsymbol{\beta}<\boldsymbol{\alpha}}m_{\boldsymbol{\alpha\beta}}(\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\boldsymbol{L}(\boldsymbol{x}))(\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\partial_t^k\boldsymbol{W}^1).\tag{44}$$

则有：

$$2|\langle\boldsymbol{A}_0\boldsymbol{g}_2^{k,\boldsymbol{\alpha}},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle|\leqslant C\|\partial_t^k(\boldsymbol{u},\Theta)\|_{|\boldsymbol{\alpha}|-1}^2+\varepsilon\|\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{u},\Theta)\|^2+C\|\partial_t^kQ\|_{|\boldsymbol{\alpha}|}^2.\tag{45}$$

由(43)-(45)得：

$$2|\langle\boldsymbol{A}_0\boldsymbol{g}^{k,\boldsymbol{\alpha}},\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle|\leqslant C\|\partial_t^k(\boldsymbol{u},\Theta)\|_{|\boldsymbol{\alpha}|-1}^2+\varepsilon\|\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{u},\Theta)\|^2+C\|\partial_t^kQ\|_{|\boldsymbol{\alpha}|}^2+C|||\boldsymbol{W}|||_s^3.\tag{46}$$

综上可得：

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle \boldsymbol{A}_0\boldsymbol{W}^1_{k,\boldsymbol{\alpha}},\boldsymbol{W}^1_{k,\boldsymbol{\alpha}}\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))+R\langle n\boldsymbol{u}_{k,\boldsymbol{\alpha}},\boldsymbol{u}_{k,\boldsymbol{\alpha}}\rangle+\langle\frac{2Rp}{(\gamma-1)\theta^2}\Theta_{k,\boldsymbol{\alpha}},\Theta_{k,\boldsymbol{\alpha}}\rangle\leqslant$$
$$\|\partial_t^k(\boldsymbol{u},\Theta,\boldsymbol{F})\|^2_{|\boldsymbol{\alpha}|-1}+2\varepsilon\|\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}(\boldsymbol{u},\Theta)\|^2+C\|\partial_t^kQ\|^2_{|\boldsymbol{\alpha}|}+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s.$$

当 ε 足够小时,由(25)可知 $\gamma>1$,则存在一个常数 $C_0>0$ 使得 $Rn-2\varepsilon\geqslant C_0$,$\frac{2Rp}{(\gamma-1)\theta^2}-2\varepsilon\geqslant C_0$ 成立,则(29)得证.

引理 4 当 $k\in N$ 且 $k\leqslant s$ 时,有

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle \boldsymbol{A}_0\partial_t^k\boldsymbol{W}^1,\partial_t^k\boldsymbol{W}^1\rangle+R(\|\partial_t^k\boldsymbol{F}\|^2+\|\partial_t^k\boldsymbol{G}\|^2))+C_0(\|\partial_t^k\boldsymbol{u}\|^2+\|\partial_t^k\Theta\|^2)\leqslant C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s. \tag{47}$$

证明 根据引理 3 的证明过程可知下面只需要证明(48)和(49)成立

$$-2R\langle\partial_t^k(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}})),n\partial_t^k\boldsymbol{u}\rangle\leqslant -R\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_t^k\boldsymbol{F}\|^2+\|\partial_t^k\boldsymbol{G}\|^2)+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s. \tag{48}$$

$$2|\langle \boldsymbol{A}_0\partial_t^k\boldsymbol{g}_2,\partial_t^k\boldsymbol{W}^1\rangle|\leqslant C|||\boldsymbol{W}|||^3_s. \tag{49}$$

由引理 3 的证明过程中对(35)右边的第二项估计可得:

$$-2R\langle\partial_t^k(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}})),n\partial_t^k\boldsymbol{u}\rangle=-2R\langle\partial_t^k\boldsymbol{F},\partial_t^k(n\boldsymbol{u})\rangle+2R\langle\partial_t^k\boldsymbol{F},\partial_t^k(n\boldsymbol{u})-n\partial_t^k\boldsymbol{u}\rangle+$$
$$2R\langle(\nabla h(\bar{n})-\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))_{k,\boldsymbol{\alpha}},n\partial_t^k\boldsymbol{u}\rangle\leqslant -R\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_t^k\boldsymbol{F}\|^2+\|\partial_t^k\boldsymbol{G}\|^2)+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s. \tag{50}$$

而$\partial_t^k\boldsymbol{g}_2=\boldsymbol{L}(\boldsymbol{x})\partial_t^k\boldsymbol{W}^1-\partial_t^k(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{W}^1)=0$,则(49)得证.

引理 5 当 $k\in N,\boldsymbol{\alpha}\in N^3$ 且 $1\leqslant|\boldsymbol{\alpha}|,k+|\boldsymbol{\alpha}|\leqslant s$ 时,有

$$\|\partial_t^k\boldsymbol{F}\|^2_{|\boldsymbol{\alpha}|-1}\leqslant C(\|\partial_t^k\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^kQ\|^2_{|\boldsymbol{\alpha}|})+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s. \tag{51}$$

$$\|\partial_t^kQ\|^2_{|\boldsymbol{\alpha}|}\leqslant C(\|\partial_t^k(\boldsymbol{u},\Theta)\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1})+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s. \tag{52}$$

证明 先证明(51),用$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}$ 作用(21)的第二个方程得:

$$\boldsymbol{F}_{k,\boldsymbol{\beta}}=-(\boldsymbol{u}_{k,\boldsymbol{\beta}}+\boldsymbol{u}_{k,\boldsymbol{\beta}}\times\bar{\boldsymbol{B}}+\boldsymbol{u}_{k+1,\boldsymbol{\beta}})-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(\boldsymbol{u}\cdot\nabla\boldsymbol{u})-\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(\boldsymbol{u}\times\boldsymbol{G})-R(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\theta\nabla Q+\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\nabla\bar{q}\Theta).\quad(|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|+1)$$
$$\|\boldsymbol{F}_{k,\boldsymbol{\beta}}\|^2\leqslant C(\|\partial_t^k\boldsymbol{u}\|^2_{|\boldsymbol{\beta}|}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\beta}|}+\|\partial_t^kQ\|^2_{|\boldsymbol{\beta}|+1})+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s.$$

所以:

$$\|\partial_t^k\boldsymbol{F}\|^2_{|\boldsymbol{\alpha}|-1}\leqslant C(\|\partial_t^k\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^kQ\|^2_{|\boldsymbol{\alpha}|})+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s.$$

下证(52),将(21)中的第二个方程写为:

$$R\theta_L\nabla Q=-(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))-(\partial_t\boldsymbol{u}+\boldsymbol{u}+R\nabla\bar{q}\Theta)+\boldsymbol{r}. \tag{53}$$

式中,$\boldsymbol{r}=-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-R\Theta\nabla Q$,$|\boldsymbol{\beta}|\leqslant|\boldsymbol{\alpha}|-1$.

用$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}}$ 作用于(53),再用$(\nabla Q)_{k,\boldsymbol{\beta}}$与它作内积得

$$R\theta_L\|(\nabla Q)_{k,\boldsymbol{\beta}}\|=\langle(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}}))_{k,\boldsymbol{\beta}},(\nabla Q)_{k,\boldsymbol{\beta}}\rangle-\langle\boldsymbol{u}_{k+1,\boldsymbol{\beta}}+\boldsymbol{u}_{k,\boldsymbol{\beta}},(\nabla Q)_{k,\boldsymbol{\beta}}\rangle-$$
$$R\langle\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(\nabla\bar{q}\Theta),(\nabla Q)_{k,\boldsymbol{\beta}}\rangle+\langle\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\boldsymbol{r},(\nabla Q)_{k,\boldsymbol{\beta}}\rangle. \tag{54}$$

下证(52):估计(54)右边的第一项得:

$$\langle\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(\boldsymbol{F}-\nabla h(\bar{n})+\boldsymbol{u}\times(\boldsymbol{G}+\bar{\boldsymbol{B}})),(\nabla Q)_{k,\boldsymbol{\beta}}\rangle\leqslant C\|\partial_t^k\boldsymbol{F}\|^2_{|\boldsymbol{\beta}|}+3\varepsilon\|(\nabla Q)_{k,\boldsymbol{\beta}}\|^2+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s$$

估计(54)右边的第二项得:

$$|\langle\boldsymbol{u}_{k,\boldsymbol{\beta}}+\boldsymbol{u}_{k+1,\boldsymbol{\beta}},(\nabla Q)_{k,\boldsymbol{\beta}}\rangle|\leqslant C(\|\partial_t^k\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1})+\varepsilon\|(\nabla Q)_{k,\boldsymbol{\beta}}\|^2. \tag{55}$$

估计(54)右边的第三项得:

$$|\langle\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(\nabla\bar{q}\Theta),(\nabla Q)_{k,\boldsymbol{\beta}}\rangle|\leqslant C\|\partial_t^k\Theta\|^2_{|\boldsymbol{\alpha}|-1}+\varepsilon\|(\nabla Q)_{k,\boldsymbol{\beta}}\|^2. \tag{56}$$

估计(54)右边的最后一项得:

$$|\langle\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\boldsymbol{r},(\nabla Q)_{k,\boldsymbol{\beta}}\rangle|\leqslant C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s.$$

综上可得:

$$R\theta_L\|(\nabla Q)_{k,\boldsymbol{\beta}}\|^2\leqslant C(\|\partial_t^k(\boldsymbol{u},\Theta)\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^kQ\|^2_{|\boldsymbol{\alpha}|})\ +C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s+5\varepsilon\|(\nabla Q)_{k,\boldsymbol{\beta}}\|^2.$$

令 $\varepsilon>0$ 足够的小,使得 $\varepsilon\leqslant R\theta_L/5$,则有:

$$\|(\nabla Q)_{k,\boldsymbol{\beta}}\|^2\leqslant C(\|\partial_t^k(\boldsymbol{u},\Theta)\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^{k+1}\boldsymbol{u}\|^2_{|\boldsymbol{\alpha}|-1}+\|\partial_t^kQ\|^2_{|\boldsymbol{\alpha}|})+C|||\boldsymbol{U}|||^2_s|||\boldsymbol{W}|||_s. \tag{57}$$

将(57)中的指数$\boldsymbol{\beta}(|\boldsymbol{\beta}|\leqslant|\boldsymbol{\alpha}|-1)$求和得到:

$$\|\partial_t^k Q\|_{|\boldsymbol{\alpha}|}^2\leqslant C(\|\partial_t^k(\boldsymbol{u},\Theta)\|_{|\boldsymbol{\alpha}|-1}^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{|\boldsymbol{\alpha}|-1}^2+\|\partial_t^k Q\|_{|\boldsymbol{\alpha}|}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{58}$$

当$C\leqslant 1$时,(52)得证.

引理 6　当$k\in N$且$k\leqslant s-1$时,我们有

$$\|\partial_t^k Q\|_1^2\leqslant C(\|\partial_t^k(\boldsymbol{u},\Theta)\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{59}$$

$$\|\partial_t^s Q\|^2\leqslant C\|\partial_t^{s-1}(\boldsymbol{u},\Theta)\|_1^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{60}$$

证明　令(52)中的$|\boldsymbol{\alpha}|=1$,可得:

$$\|\partial_t^k Q\|_1^2\leqslant C(\|\partial_t^k(\boldsymbol{u},\Theta)\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s.$$

最后,由(21)的第一个方程和引理 2,容易证得(60).

引理 7

$$|||\boldsymbol{F}|||_{s-1}\leqslant C|||\boldsymbol{U}|||_s. \tag{61}$$

$$|||\partial_t\boldsymbol{G}|||_{s-2}+|||\nabla\boldsymbol{G}|||_{s-2}\leqslant C|||\boldsymbol{U}|||_s. \tag{62}$$

证明见[15].

命题 3　当$k\in N,\boldsymbol{\alpha}\in N^3$且$1\leqslant|\boldsymbol{\alpha}|,k+|\boldsymbol{\alpha}|\leqslant s$,有

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}}(\langle\boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\beta}}^1,\boldsymbol{W}_{k,\boldsymbol{\beta}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\beta}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\beta}}\|^2))+C_0\|\partial_t^k U\|_{|\boldsymbol{\alpha}|}^2\leqslant$$
$$(\|\partial_t^k\boldsymbol{U}\|_{|\boldsymbol{\alpha}|-1}^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{|\boldsymbol{\alpha}|-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{63}$$

证明　由引理 3 和引理 5 可得:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\langle\boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\beta}}^1,\boldsymbol{W}_{k,\boldsymbol{\beta}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\beta}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\beta}}\|^2))+C_0(\|\boldsymbol{u}_{k,\boldsymbol{\beta}}\|^2+\|\Theta_{k,\boldsymbol{\beta}}\|^2)\leqslant$$
$$C(\|\partial_t^k(\boldsymbol{u},\Theta)\|_{|\boldsymbol{\beta}|-1}^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{|\boldsymbol{\beta}|-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s.$$

令$k\geqslant 0$固定,对上面不等式中的指数$\boldsymbol{\beta}(|\boldsymbol{\beta}|\leqslant|\boldsymbol{\alpha}|)$求和,其中$\boldsymbol{\beta}$满足$1\leqslant|\boldsymbol{\beta}|,k+|\boldsymbol{\beta}|\leqslant s$,再结合引理 4 得到(63).

命题 4　当W_T足够小时,有

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\boldsymbol{\beta}\leqslant\boldsymbol{\alpha}}(\langle\boldsymbol{A}_0\partial_t^s\boldsymbol{W}^1,\partial_t^s\boldsymbol{W}^1\rangle+R(\|\partial_t^s\boldsymbol{F}\|^2+\|\partial_t^s\boldsymbol{G}\|^2))+C_0\|\partial_t^s\boldsymbol{U}\|^2\leqslant C\|\partial_t^{s-1}\boldsymbol{U}\|_1^2+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{64}$$

4　存在性定理证明

定理 1　在命题 2 的假设下,存在与$t(t>0)$无关的常数$\delta_0>0,C>0$,若$\|(q-\bar{q},\boldsymbol{u}_0,\theta-\theta_L,\boldsymbol{E}_0-\bar{\boldsymbol{E}},\boldsymbol{B}_0-\bar{\boldsymbol{B}})\|\leqslant\delta_0$成立,则问题(1)-(3)存在唯一的整体光滑解$(Q,\boldsymbol{u},\Theta,\boldsymbol{E},\boldsymbol{B})$满足:

$$(q(t)-\bar{q},\boldsymbol{u}(t),\theta(t)-\theta_L,\boldsymbol{E}(t)-\bar{\boldsymbol{E}},\boldsymbol{B}(t)-\bar{\boldsymbol{B}})\in\bigcap_{k=0}^{s}C^k([0,+\infty];H^{s-k}(T^3)). \tag{65}$$

$$\sup_{t\geqslant 0}|||q(t)-\bar{q},\boldsymbol{u}(t),\theta(t)-\theta_L,\boldsymbol{E}(t)-\bar{\boldsymbol{E}},\boldsymbol{B}(t)-\bar{\boldsymbol{B}}|||_s\leqslant C\|(q_0-\bar{q},\boldsymbol{u}_0,\theta_0-\theta_L,\boldsymbol{E}_0-\bar{\boldsymbol{E}},\boldsymbol{B}_0-\bar{\boldsymbol{B}})\|_s. \tag{66}$$

$$\int_0^{+\infty}(|||(q(t)-\bar{q},\boldsymbol{u}(t),\theta(t)-\theta_L)|||_s+|||\boldsymbol{E}(t)-\bar{\boldsymbol{E}}|||_{s-1}+|||\partial_t\boldsymbol{B}(t)|||_{s-2}+|||\nabla_x\boldsymbol{B}(t)|||_{s-2})\mathrm{d}t\leqslant$$
$$C\|(q_0-\bar{q},\boldsymbol{u}_0,\theta_0-\theta_L,\boldsymbol{E}_0-\bar{\boldsymbol{E}},\boldsymbol{B}_0-\bar{\boldsymbol{B}})\|_s. \tag{67}$$

此外,有

$$\lim_{t\to+\infty}|||q(t)-\bar{q},\boldsymbol{u}(t),\theta(t)-\theta_L|||_{s-1}=0. \tag{68}$$

$$\lim_{t\to+\infty}|||\boldsymbol{E}(t)-\bar{\boldsymbol{E}}|||_{s-1}=0. \tag{69}$$

$$\lim_{t\to+\infty}(|||\partial_t\boldsymbol{B}(t)|||_{s-2}+|||\nabla_x\boldsymbol{B}(t)|||_{s-2})=0. \tag{70}$$

证明　首先,对于任何固定的指数$k\in N$且$k\leqslant s-1$,对(63)的空间导数$\boldsymbol{\alpha}(1\leqslant|\boldsymbol{\alpha}|\leqslant s-k)$进行归纳,从$|\boldsymbol{\alpha}|=1$到$|\boldsymbol{\alpha}|=s-k$进行归纳,当$|\boldsymbol{\alpha}|\geqslant 2$时,(63)右边的项$\|\partial_t^k\boldsymbol{U}\|_{|\boldsymbol{\alpha}|-1}$能够被$\|\partial_t^k\boldsymbol{U}\|_{|\boldsymbol{\alpha}|}$控制,将(63)左边的项与一个合适的正常数相乘得:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\boldsymbol{\alpha}|\leqslant s-k}a_{k,\boldsymbol{\alpha}}(\langle\boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))+\|\partial_t^k\boldsymbol{U}\|_{s-k}^2\leqslant$$

$$C(\|\partial_t^k \boldsymbol{U}\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|_{s-k-1}^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{71}$$

其中 $a_{k,\boldsymbol{\alpha}}>0(k\leqslant s-1,1\leqslant|\boldsymbol{\alpha}|\leqslant s-k)$ 是常数.

下面,对 k 从 $k=s$ 到 $k=0$ 作一个归纳,当 $k=s$ 时,相应的估计由(64)给出,当 $k=s-1$ 时,由(71)可得:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\boldsymbol{\alpha}|\leqslant 1} a_{s-1,\boldsymbol{\alpha}}(\langle \boldsymbol{A}_0\boldsymbol{W}_{s-1,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{s-1,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{s-1,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{s-1,\boldsymbol{\alpha}}\|^2))+\|\partial_t^{s-1}\boldsymbol{U}\|_1^2\leqslant$$
$$(\|\partial_t^{s-1}\boldsymbol{U}\|^2+\|\partial_t^s\boldsymbol{U}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{72}$$

显然,(72)左边的项乘以一个合适的常数修正之后可以控制(64)右边的项 $\|\partial_t^{s-1}\boldsymbol{U}\|^2$,由前面的步骤可知 $\|\partial_t^{k+1}\boldsymbol{u}\|_{s-k-1}^2$ 能够被 $\|\partial_t^k\boldsymbol{U}\|_{s-k}^2$ 控制,用这种方法对 k 进行归纳后得:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\boldsymbol{\alpha}|\leqslant s} a_{k,\boldsymbol{\alpha}}(\langle \boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))+\sum_{k=0}^{s}\|\partial_t^k\boldsymbol{U}\|_{s-k}^2\leqslant$$
$$C\sum_{k=0}^{s-1}(\|\partial_t^k\boldsymbol{U}\|^2+\|\partial_t^{k+1}\boldsymbol{u}\|^2)+C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}|||_s. \tag{73}$$

其中正常数 $a_{k,\boldsymbol{\alpha}}$ 是以(71)为基础修正得到的,由(48),(50)和(73)可知 $\sum\limits_{k=0}^{s}\|\partial_t^k\boldsymbol{U}\|_{s-k}^2$ 与 $|||\boldsymbol{U}|||_s^3$ 等价,对 $a_{k,\boldsymbol{\alpha}}$ 再次修正得到:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\boldsymbol{\alpha}|\leqslant s} a_{k,\boldsymbol{\alpha}}(\langle \boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))+|||\boldsymbol{U}(t,\cdot)|||_s^2\leqslant C|||\boldsymbol{U}|||_s^2|||\boldsymbol{W}||| \tag{74}$$

因为 W_T 足够的小,所以进一步得到:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\boldsymbol{\alpha}|\leqslant s} a_{k,\boldsymbol{\alpha}}(\langle \boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))+|||\boldsymbol{U}(t,\cdot)|||_s^2\leqslant 0. \tag{75}$$

结合引理 7 再次修正常数 $a_{k,\boldsymbol{\alpha}}$ 得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\boldsymbol{\alpha}|\leqslant s} a_{k,\boldsymbol{\alpha}}(\langle \boldsymbol{A}_0\boldsymbol{U}_{k,\boldsymbol{\alpha}},\boldsymbol{U}_{k,\boldsymbol{\alpha}}\rangle+\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2)+|||\boldsymbol{U}|||_s^2+|||\boldsymbol{F}|||_{s-1}^2+(|||\partial_t\boldsymbol{G}|||_{s-2}+|||\nabla\boldsymbol{G}|||_{s-2})\leqslant 0$$

因为 $\frac{\mathrm{d}}{\mathrm{d}t}\sum\limits_{k+|\boldsymbol{\alpha}|\leqslant s} a_{k,\boldsymbol{\alpha}}(\langle \boldsymbol{A}_0\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1,\boldsymbol{W}_{k,\boldsymbol{\alpha}}^1\rangle+R(\|\boldsymbol{F}_{k,\boldsymbol{\alpha}}\|^2+\|\boldsymbol{G}_{k,\boldsymbol{\alpha}}\|^2))$ 与 $|||\boldsymbol{W}|||_s^2$ 等价,所以将(75)在 $[0,t]$ 上积分可得:

$$|||\boldsymbol{W}(t)|||_s^2+\int_0^t(|||\boldsymbol{W}(\tau)|||_s^2+R|||\boldsymbol{F}|||_{s-1}^2+R|||\partial_t\boldsymbol{G}|||_{s-2}+R|||\nabla\boldsymbol{G}|||_{s-2})\mathrm{d}\tau\leqslant C|||\boldsymbol{W}(0)|||_s^2. \tag{76}$$

方程组(21)可以写成:

$$\partial_t\boldsymbol{W}=f(x,\boldsymbol{W},\partial_x\boldsymbol{W}),\ f\text{ 是一个光滑函数且 } f(x,0,0)=0 \tag{77}$$

由引理 1 可知:

$$|||\boldsymbol{W}(0)|||_s\leqslant C\|\boldsymbol{W}_0\|_s. \tag{78}$$

(71)-(78)证明了整体解的存在性和(65)-(67).

此外,由(76)可知,当 $k+|\boldsymbol{\beta}|\leqslant s-1$ 时,有

$$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\boldsymbol{W}\in L^\infty(0,+\infty;L^2(T^3)),\ \partial_t(\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\boldsymbol{W})\in L^\infty(0,+\infty;L^2(T^3)).$$

则

$$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\boldsymbol{W}\in W^{1,\infty}(0,+\infty;L^2(T^3)),\ \forall k+|\boldsymbol{\beta}|\leqslant s-1.$$

所以:

$$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(q-\bar{q},\boldsymbol{u},\theta-\theta_L,\boldsymbol{E}-\bar{\boldsymbol{E}})\in L^2(0,+\infty;L^2(T^3)),\ \forall k+|\boldsymbol{\alpha}|\leqslant s-1.$$

得到:

$$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}(q-\bar{q},\boldsymbol{u},\theta-\theta_L,\boldsymbol{E}-\bar{\boldsymbol{E}})\in L^2(0,+\infty;L^2(T^3))\cap W^{1,\infty}(0,+\infty,L^2(T^3))\ \forall k+|\boldsymbol{\beta}|\leqslant s-1.$$

则(68)和(69)得证. 同理,因为 $\bar{\boldsymbol{B}}$ 是一个常数向量,则有

$$\partial_t^k\partial_{\boldsymbol{x}}^{\boldsymbol{\beta}}\boldsymbol{B}\in L^2(0,+\infty;L^2(T^3))\cap W^{1,\infty}(0,+\infty,L^2(T^3)),\ \forall 1\leqslant k+|\boldsymbol{\beta}|\leqslant s-1.$$

所以(70)得证.

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