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Von Neumann 迹的不等式注记

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[**摘要**]　通过矩阵分块,利用矩阵特征值与奇异值的性质,研究 Von Neumann 迹的不等式,推广了相关文献矩阵乘积之迹的不等式,并对有关文献作了补充.

[**关键词**]　Von Neumann 不等式,特征值,奇异值,迹,Frobenius 范数

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A Note on Von Neumann's Trace Inequality

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Abstract:In this paper,the inequality of Von Neumann trace was studied by using the properties of the matrix divided into blocks,singular value and eigenvalue of the matrix. As a result,the inequalities of the matrix product trace were improved under the certain conditions. Besides,the established conclusions were extended.

Key words:Von Neumann inequality,eigenvalue,singular value,trace,Frobenius norm

本文中,设 a,b 为实数,Re x 表示复数 $x=a+\mathrm{i}b$ 的实部,即 Re $x=a$,Im $x=b$ 表示复数 x 的虚部. $\boldsymbol{M}^{m\times n}$ 表示 m 行 n 列复矩阵的集合. $\boldsymbol{I}$ 为单位矩阵,$\boldsymbol{A}^{\mathrm{H}}$ 表示 $\boldsymbol{A}$ 的共轭转置矩阵. 如果 $\boldsymbol{A}=\boldsymbol{A}^{\mathrm{H}}$,则称 $\boldsymbol{A}$ 为 Herimite 阵. 设 $\lambda_i(\boldsymbol{A})(i=1,\cdots,n)$ 为 n 阶矩阵 $\boldsymbol{A}$ 的特征值,当 $\boldsymbol{A}$ 为 Herimite 阵,且 $\lambda_i(\boldsymbol{A})\geqslant 0(i=1,\cdots,n)$ 时,则 $\boldsymbol{A}$ 称为 Herimite 半正定阵. $\sigma_i(\boldsymbol{A})$ 表示 $\boldsymbol{A}$ 的奇异值($i=1,\cdots,n$). 用 diag$(d_1,\cdots,d_n)$ 表示对角元素为 $d_1,\cdots,d_n$ 的对角矩阵. 记 $\boldsymbol{M}^{m\times n}$ 上的矩阵 Frobenius 范数为

$$\|\boldsymbol{A}\|_F=(\mathrm{tr}\,\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})^{1/2}=\Big(\sum_{i,j}|a_{ij}|^2\Big)^{1/2},\quad \boldsymbol{A}=(a_{ij})\in\boldsymbol{M}^{m\times n}.$$

这里 tr $\boldsymbol{A}$ 表示矩阵 $\boldsymbol{A}$ 的迹.

关于矩阵的迹,1937 年,Von Neumann 提出了著名不等式[1]

$$|\mathrm{tr}(\boldsymbol{AB})|\leqslant\sum_{i=1}^{n}\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),$$

式中,$\boldsymbol{A},\boldsymbol{B}$ 是 $n\times n$ 阶复矩阵. 此不等式不仅在数学分支如数值代数中具有重要意义,而且在信号处理、通信工程、系统工程等学科中也有着广泛的应用,国内外学者对此不等式的研究十分活跃[1-15].

1975 年,Mirsky L 推广了 Von Neumann 迹的不等式,用双随机矩阵的性质在文献[2-3]中证明了如下等式

$$\sup_{\boldsymbol{U},\boldsymbol{V}}\mathrm{Re}\ \mathrm{tr}\,\boldsymbol{UAVB}=\sup_{\boldsymbol{U},\boldsymbol{V}}|\mathrm{tr}\,\boldsymbol{UAVB}|=\sum_{i=1}^{n}\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),\tag{1}$$

式中,$\boldsymbol{U},\boldsymbol{V}$ 为酉矩阵,$\boldsymbol{A},\boldsymbol{B}$ 为 n 阶复矩阵.

1979 年,Marshall 和 Olkin 在文献[4]中利用控制不等式的性质对式(1)给出重新证明并作了进一步推广.

2003 年,Ben-Isral 和 Greville 给出不等式[5]

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$$\mathrm{Re}\ \mathrm{tr}\ (\boldsymbol{HW})\leqslant \mathrm{tr}\ \boldsymbol{H},\tag{2}$$

式中,$\boldsymbol{H}$ 为任意的 Herimite 半正定阵,$\boldsymbol{W}$ 为酉矩阵.

设 $\boldsymbol{A}\in \boldsymbol{M}^{m\times n}$,则 $\boldsymbol{A}$ 的奇异值分解为[6]

$$\boldsymbol{A}=\boldsymbol{P}(\boldsymbol{\Sigma}_1,\boldsymbol{O})\boldsymbol{Q},\tag{3}$$

式中,$\boldsymbol{\Sigma}_1=\mathrm{diag}(\sigma_1(\boldsymbol{A}),\cdots,\sigma_m(\boldsymbol{A}))$,$\sigma_1(\boldsymbol{A})\geqslant\cdots\geqslant\sigma_m(\boldsymbol{A})\geqslant 0$. $\boldsymbol{O}$ 为 $m\times(n-m)$ 零矩阵. $\boldsymbol{P}$,$\boldsymbol{Q}$ 分别为 m,n 阶酉矩阵.

设 $\boldsymbol{B}\in \boldsymbol{M}^{n\times m}$,则 $\boldsymbol{B}$ 的奇异值分解为[6]

$$\boldsymbol{B}=\boldsymbol{R}\begin{pmatrix}\Sigma_2\\ O\end{pmatrix}\boldsymbol{S},\tag{4}$$

式中,$\boldsymbol{\Sigma}_2=\mathrm{diag}(\sigma_1(\boldsymbol{B}),\cdots,\sigma_m(\boldsymbol{B}))$,$\sigma_1(\boldsymbol{B})\geqslant\cdots\geqslant\sigma_m(\boldsymbol{B})\geqslant 0$. $\boldsymbol{R}$,$\boldsymbol{S}$ 分别为 n,m 阶酉矩阵.

事实上,因为对于 $m\times n$ 矩阵 $\boldsymbol{A}$ 和 $n\times m$ 矩阵 $\boldsymbol{B}$,总有 $\mathrm{tr}(\boldsymbol{AB})=\mathrm{tr}(\boldsymbol{BA})$,所以不妨设 $m\leqslant n$,故本文中的奇异值分解假定 $m\leqslant n$.

2007 年,Komaroff N 通过分解式(3)与(4)就 n 阶复方阵 $\boldsymbol{A}$、$\boldsymbol{B}$ 对 Von Neumann 不等式给出进一步的推广,获得如下不等式[7]

$$\mathrm{Re}\ \mathrm{tr}\ (\boldsymbol{VAWB})\leqslant \sum_{i=1}^{n}|\mathrm{Re}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),\tag{5}$$

$$\mathrm{Im}\ \mathrm{tr}(\boldsymbol{VAWB})\leqslant \sum_{i=1}^{n}|\mathrm{Im}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),\tag{6}$$

式中,$\boldsymbol{T}=\boldsymbol{VPQWRS}$. 显然,当 $\boldsymbol{V}=\boldsymbol{B}=\boldsymbol{I}_n$ 时,式(5)与(6)分别为

$$\mathrm{Re}\ \mathrm{tr}\ (\boldsymbol{AW})\leqslant \sum_{i=1}^{n}|\mathrm{Re}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A}),\tag{7}$$

$$\mathrm{Im}\ \mathrm{tr}(\boldsymbol{AW})\leqslant \sum_{i=1}^{n}|\mathrm{Im}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A}),\tag{8}$$

式中,$\boldsymbol{T}=\boldsymbol{PQW}$.

本文讨论 $\boldsymbol{A}\in \boldsymbol{M}^{m\times n}$,$\boldsymbol{B}\in \boldsymbol{M}^{n\times m}$的一般情形,推广式(5)-(8),给出矩阵乘积之迹的另一表达式. 所获结论在数值代数等领域将会有重要应用.

本文需要如下引理.

引理 1[8]　设 $\boldsymbol{A}$、$\boldsymbol{B}$ 为 n 阶矩阵,则 $\boldsymbol{AB}$ 与 $\boldsymbol{BA}$ 有相同的特征值和迹.

1　主要结论

定理 1　设 $\boldsymbol{A}\in \boldsymbol{M}^{m\times n}$,$\boldsymbol{B}\in \boldsymbol{M}^{n\times m}$,$\boldsymbol{W}$ 为 n 阶酉矩阵,$\boldsymbol{V}$ 为 m 阶酉矩阵,$\boldsymbol{A}$、$\boldsymbol{B}$ 的奇异值分解如式(3)、(4)所示,$m\leqslant n$,则

$$\mathrm{Re}\ \mathrm{tr}\ (\boldsymbol{VAWB})\leqslant \sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\tilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),\tag{9}$$

$$\mathrm{Im}\ \mathrm{tr}(\boldsymbol{VAWB})\leqslant \sum_{i=1}^{m}|\mathrm{Im}\ \lambda_i(\tilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),\tag{10}$$

式中,$\tilde{\boldsymbol{T}}=\boldsymbol{QWR}\begin{pmatrix}\boldsymbol{SVP} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_{n-m}\end{pmatrix}$.

证　$\boldsymbol{V}$,$\boldsymbol{P}$,$\boldsymbol{S}$ 如分解式(3)与(4)所设,令 $\tilde{\boldsymbol{V}}=\begin{pmatrix}\boldsymbol{V} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_{n-m}\end{pmatrix}$,$\tilde{\boldsymbol{P}}=\begin{pmatrix}\boldsymbol{P} & \boldsymbol{O}\\ \boldsymbol{O} & I_{n-m}\end{pmatrix}$,$\tilde{\boldsymbol{S}}=\begin{pmatrix}\boldsymbol{S} & \boldsymbol{O}\\ \boldsymbol{O} & I_{n-m}\end{pmatrix}$,则 $\tilde{\boldsymbol{V}}$,$\tilde{\boldsymbol{P}}$,$\tilde{\boldsymbol{S}}$ 为 n 阶酉矩阵. 设 $\tilde{\boldsymbol{A}}=\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{O}\end{pmatrix}\in \boldsymbol{M}^{n\times n}$,$\tilde{\boldsymbol{B}}=(\boldsymbol{B},\boldsymbol{O})\in \boldsymbol{M}^{n\times n}$,$\boldsymbol{O}$ 为相应的零矩阵. 则由分解式(3)与(4)得

$$\tilde{\boldsymbol{A}}=\begin{pmatrix}\boldsymbol{P}(\boldsymbol{\Sigma}_1,\boldsymbol{O})\boldsymbol{Q}\\ \boldsymbol{O}\end{pmatrix}\triangleq\begin{pmatrix}\boldsymbol{P\Sigma}_1 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}\boldsymbol{Q}=\tilde{\boldsymbol{P}}\begin{pmatrix}\boldsymbol{\Sigma}_1 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}\boldsymbol{Q},$$

$$\tilde{\boldsymbol{B}}=\left(\boldsymbol{R}\begin{pmatrix}\boldsymbol{\Sigma}_2\\ \boldsymbol{O}\end{pmatrix}\boldsymbol{S},\boldsymbol{O}\right)\triangleq\boldsymbol{R}\begin{pmatrix}\boldsymbol{\Sigma}_2\boldsymbol{S} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}=\boldsymbol{R}\begin{pmatrix}\boldsymbol{\Sigma}_2 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}\tilde{\boldsymbol{S}},$$

且 $\sigma_i(\widetilde{\boldsymbol{A}})=\sigma_i(\boldsymbol{A})$,$\sigma_i(\widetilde{\boldsymbol{B}})=\sigma_i(\boldsymbol{B})$. 令 $\widetilde{\boldsymbol{T}}_1=\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{P}}\boldsymbol{QWR}\widetilde{\boldsymbol{S}}$,则 $\widetilde{\boldsymbol{T}}_1$ 为酉矩阵. 由引理 1 得

$$\lambda(\widetilde{\boldsymbol{T}}_1)=\lambda(\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{P}}\boldsymbol{QWR}\widetilde{\boldsymbol{S}})=\lambda(\boldsymbol{QWR}\widetilde{\boldsymbol{S}}\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{P}})=\lambda\left[\boldsymbol{QWP}\begin{pmatrix}\boldsymbol{SVP} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_{n-m}\end{pmatrix}\right]=\lambda(\widetilde{T}).$$

于是由式(5),我们有

$$\mathrm{Re\ tr}\ (\boldsymbol{VAWB})=\mathrm{tr}\begin{pmatrix}\boldsymbol{VAWB} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}=\mathrm{Re\ tr}\left[\begin{pmatrix}\boldsymbol{V} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_{n-m}\end{pmatrix}\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{O}\end{pmatrix}\boldsymbol{W}(\boldsymbol{B}\quad \boldsymbol{O})\right]=\mathrm{Re\ tr}\ (\widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{A}}\boldsymbol{W}\widetilde{\boldsymbol{B}})\leqslant$$

$$\sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{T}}_1)|\sigma_i(\widetilde{\boldsymbol{A}})\sigma_i(\widetilde{\boldsymbol{B}})=\sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}).$$

即不等式(9)成立,同理可证不等式(10).

注 1 同定理 1 的证明,当 $m\geqslant n$ 时,有

$$\mathrm{Re\ tr}\ (\boldsymbol{VAWB})\leqslant\sum_{i=1}^{n}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),$$

$$\mathrm{Im\ tr}(\boldsymbol{VAWB})\leqslant\sum_{i=1}^{n}|\mathrm{Im}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}),$$

式中,$\widetilde{\boldsymbol{T}}=\boldsymbol{SVP}\begin{pmatrix}\boldsymbol{QWR} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_{m-n}\end{pmatrix}$.

注 2 设 $\boldsymbol{H}$ 为 n 阶 Hermite 半正定阵,则存在 n 阶矩阵 $\boldsymbol{A}\in\boldsymbol{M}^{n\times n}$,使 $\boldsymbol{H}=\boldsymbol{AA}^{\mathrm{H}}$. 注意到

$$|\mathrm{Re}\ \lambda_i(\boldsymbol{W})|\leqslant|\lambda_i(\boldsymbol{W})|\leqslant 1.$$

我们有

$$\mathrm{Re\ tr}\ (\boldsymbol{WH})\leqslant\sum_{i=1}^{n}|\mathrm{Re}\ \lambda_i(\boldsymbol{W})|\lambda_i(\boldsymbol{AA}^{\mathrm{H}})\leqslant\sum_{i=1}^{n}\lambda_i(\boldsymbol{H})=\mathrm{tr}\ \boldsymbol{H}.$$

此即不等式(2).

定理 2 设 $\boldsymbol{A}\in\boldsymbol{M}^{m\times n}$,$\boldsymbol{W}\in\boldsymbol{M}^{n\times m}$为列酉阵,即 $\boldsymbol{W}^H\boldsymbol{W}=\boldsymbol{I}_m$,则

$$\mathrm{Re\ tr}\ (\boldsymbol{AW})\leqslant\sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A}),\tag{11}$$

$$\mathrm{Im\ tr}(\boldsymbol{AW})\leqslant\sum_{i=1}^{m}|\mathrm{Im}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A}),\tag{12}$$

式中,$\widetilde{\boldsymbol{T}}=(\boldsymbol{QWP},\boldsymbol{QW}^{\perp})$,$\boldsymbol{W}^{\mathrm{H}}\boldsymbol{W}^{\perp}=\boldsymbol{O}$,$(\boldsymbol{W}^{\perp})^{\mathrm{H}}\boldsymbol{W}^{\perp}=\boldsymbol{I}_{n-m}$.

证 令 $\widetilde{\boldsymbol{A}}=\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{O}\end{pmatrix}\in\boldsymbol{M}^{m\times n}$,$\boldsymbol{O}$ 为$(n-m)\times n$ 阶零矩阵. 令 $\widetilde{\boldsymbol{W}}=(\boldsymbol{W},\boldsymbol{W}^{\perp})$,其中 $\boldsymbol{W}^{\perp}$ 为 $n\times(n-m)$ 阶列酉阵,且 $\boldsymbol{W}^{\mathrm{H}}\boldsymbol{W}^{\perp}=\boldsymbol{O}$,则 $\widetilde{\boldsymbol{W}}$ 为 n 阶酉矩阵. 由 A 的奇异值分解式(3)得

$$\widetilde{\boldsymbol{A}}=\widetilde{\boldsymbol{P}}\begin{pmatrix}\boldsymbol{\Sigma}_1 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}\boldsymbol{Q}.$$

故由式(7)与引理 1 以及

$$\lambda(\widetilde{\boldsymbol{T}})=\lambda\left[\boldsymbol{Q}(\boldsymbol{W},\boldsymbol{W}^{\perp})\begin{pmatrix}\boldsymbol{P} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_{n-m}\end{pmatrix}\right]=\lambda(\boldsymbol{Q}\widetilde{\boldsymbol{W}}\widetilde{\boldsymbol{P}})=\lambda(\widetilde{\boldsymbol{P}}\boldsymbol{Q}\widetilde{\boldsymbol{W}}).$$

我们有

$$\mathrm{Re\ tr}\ (\boldsymbol{AW})=\mathrm{Re\ tr}\begin{pmatrix}\boldsymbol{AW} & \boldsymbol{AW}^{\perp}\\ \boldsymbol{O} & \boldsymbol{O}\end{pmatrix}=\mathrm{Re\ tr}\left(\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{O}\end{pmatrix}(\boldsymbol{W}\quad \boldsymbol{W}^{\perp})\right)=\mathrm{Re\ tr}\ \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{W}}\leqslant$$

$$\sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{P}}\boldsymbol{Q}\widetilde{\boldsymbol{W}})|\sigma_i(\boldsymbol{A})=\sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A}).$$

同理可证式(12).

推论 1 设 $\boldsymbol{A}\in\boldsymbol{M}^{m\times n}$,$\boldsymbol{V}$ 为任 m 阶酉阵,$\boldsymbol{W}$ 为 $n\times m$ 阶列酉阵,即 $\boldsymbol{W}^{\mathrm{H}}\boldsymbol{W}=\boldsymbol{I}_m$ 则

$$\mathrm{Re\ tr}\ (\boldsymbol{VAW})\leqslant\sum_{i=1}^{m}|\mathrm{Re}\ \lambda_i(\widetilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A}),$$

$$\mathrm{Im}\ \mathrm{tr}(\boldsymbol{VAW}) \leqslant \sum_{i=1}^{m} |\mathrm{Im}\ \lambda_i(\tilde{\boldsymbol{T}})|\sigma_i(\boldsymbol{A}),$$

式中,$\tilde{\boldsymbol{T}}=(\boldsymbol{QWVP},\boldsymbol{QW}^{\perp})$,$\boldsymbol{W}^{\perp}$ 为列酉阵,且 $\boldsymbol{W}^{\mathrm{H}}\boldsymbol{W}^{\perp}=\boldsymbol{O}$.

证　由引理 1,我们有 $\mathrm{tr}(\boldsymbol{VAW})=\mathrm{tr}(\boldsymbol{AWV})$. 因为 $\boldsymbol{W}$ 为 $n\times m$ 阶列酉阵,$\boldsymbol{V}$ 为 m 阶酉阵. 所以 $\boldsymbol{WV}$ 为 $n\times m$ 阶列酉阵. 于是由定理 2 即得所证.

定理 3　设 $\boldsymbol{A}$、$\boldsymbol{B}$ 为 n 阶复矩阵,$\boldsymbol{V}$、$\boldsymbol{W}$ 为 n 阶酉阵,并设 $\boldsymbol{A}$、$\boldsymbol{B}$ 的奇异值分解分别为

$$\boldsymbol{A}=\boldsymbol{P\Sigma}_1\boldsymbol{Q},\boldsymbol{B}=\boldsymbol{R\Sigma}_2\boldsymbol{S},$$

式中,$\boldsymbol{P}$,$\boldsymbol{Q}$,$\boldsymbol{R}$,$\boldsymbol{S}$ 为 n 阶酉矩阵,

$$\boldsymbol{\Sigma}_1=\mathrm{diag}(\sigma_1(\boldsymbol{A}),\cdots\sigma_i,(\boldsymbol{A})),\boldsymbol{\Sigma}_2=\mathrm{diag}(\sigma_1(\boldsymbol{B}),\cdots,\sigma_i(\boldsymbol{B})).$$

令 $\boldsymbol{T}=\boldsymbol{VRSWQ}^{\mathrm{H}}\boldsymbol{P}^{\mathrm{H}}$. 则

$$\|\boldsymbol{A}-\boldsymbol{VBW}\|_F \geqslant \sqrt{\sum_{i=1}^{n}[\sigma_i^2(\boldsymbol{A})-2|\mathrm{Re}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B})+\sigma_i^2(\boldsymbol{B})]}. \tag{13}$$

证　由式(5)有

$$\|\boldsymbol{A}-\boldsymbol{VBW}\|_F^2=\mathrm{tr}[(\boldsymbol{A}-\boldsymbol{VBW})^{\mathrm{H}}(\boldsymbol{A}-\boldsymbol{VBW})]=\mathrm{tr}[\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}+\boldsymbol{W}^{\mathrm{H}}\boldsymbol{B}^{\mathrm{H}}\boldsymbol{VBW}-((\boldsymbol{VBW})^{\mathrm{H}}\boldsymbol{A}+\boldsymbol{A}^{\mathrm{H}}(\boldsymbol{VBW}))]=$$

$$\mathrm{tr}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}+\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B})-\mathrm{tr}[(\boldsymbol{VBW})^{\mathrm{H}}\boldsymbol{A}+\boldsymbol{A}^{\mathrm{H}}(\boldsymbol{VBW})]=\sum_{i=1}^{n}\lambda_i(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A})+\sum_{i=1}^{n}\lambda_i(\boldsymbol{B}^{\mathrm{H}}\boldsymbol{B})-2\mathrm{Re}\ \mathrm{tr}\ (\boldsymbol{A}^{\mathrm{H}}(\boldsymbol{VBW}))\geqslant$$

$$\sum_{i=1}^{n}\sigma_i^2(\boldsymbol{A})+\sum_{i=1}^{n}\sigma_i^2(\boldsymbol{B})-2\sum_{i=1}^{n}|\mathrm{Re}\ \lambda(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}).$$

注意到$|\mathrm{Re}\ \lambda(\boldsymbol{T})|\leqslant 1$,因而有

$$\sum_{i=1}^{n}(\sigma_i^2(\boldsymbol{A})-2|\mathrm{Re}\ \lambda(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B})+\sigma_i^2(\boldsymbol{B}))\geqslant\sum_{i=1}^{n}(\sigma_i(\boldsymbol{A})-\sigma_i(\boldsymbol{B}))^2\geqslant 0.$$

所以

$$\|\boldsymbol{A}-\boldsymbol{VBW}\|_F^2\geqslant\sum_{i=1}^{n}[\sigma_i^2(\boldsymbol{A})-2|\mathrm{Re}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B})+\sigma_i^2(\boldsymbol{B})].$$

上式两边开方即得证.

注 3　令 $\boldsymbol{A}_\sigma=\mathrm{diag}(\sigma_1(\boldsymbol{A}),\cdots,\sigma_n(\boldsymbol{A}))$,$\boldsymbol{B}_\sigma=\mathrm{diag}(\sigma_1(\boldsymbol{B}),\cdots,\sigma_n(\boldsymbol{B}))$,则由定理 3 的证明我们有

$$\|\boldsymbol{A}-\boldsymbol{VBW}\|_F\geqslant\sqrt{\sum_{i=1}^{n}[\sigma_i^2(\boldsymbol{A})-2|\mathrm{Re}\ \lambda_i(\boldsymbol{T})|\sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B})+\sigma_i^2(\boldsymbol{B})]}\geqslant$$

$$\sqrt{\sum_{i=1}^{n}|\sigma_i(\boldsymbol{A})-\sigma_i(\boldsymbol{B})|^2}=\|\boldsymbol{A}_\sigma-\boldsymbol{B}_\sigma\|_F.$$

而不等式 $\|\boldsymbol{A}-\boldsymbol{VBW}\|_F\geqslant\|\boldsymbol{A}_\sigma-\boldsymbol{B}_\sigma\|_F$ 是文献[15]定理 2 中酉不变范数取 Frobenius 范数之情形.

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