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Complex Chooser Option Pricing for Continuous O-U Process

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Abstract:We consider the complex chooser option pricing problem when the stock price follows a continuous generalized exponential Ornstein-Uhlenbeck process model. We suppose that risk interest rate,the expected return rate and volatility of the stock price are functions of time. We adopt the martingale approach to price the complex chooser option,the analytical pricing formula of the complex chooser options is derived. We also give the actuarial methods for pricing the complex chooser option and we derive the analytical pricing formula of the complex chooser options. Some conclusions are also given.

Key words:complex chooser option pricing,O-U process,martingale,measure transforms,insurance actuarial

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连续 O-U 过程下的欧式复杂任选期权定价

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[**摘要**] 研究股票价格服从连续广义指数 O-U 过程模型下的复杂任选期权的定价问题. 假设无风险利率、波动率都是时间的函数,首先采用鞅方法得到复杂任选期权的价格公式,然后用保险精算的方法,给出了复杂任选期权在任意时刻 t 的价格.

[**关键词**] 复杂任选期权,O-U 过程,鞅,测度变换,保险精算

1 Martingale Approach to Complex Chooser Option Pricing

We assume the complex chooser option with the chooser time T and the underlying assets price is $S(t)$ at time t. The call option price with maturity at time T_1 and strike price K_1 is $C_{T_1,K_1}(t,S(t))$ at time t,the put option price with maturity at time T_2 and strike price K_2 is $P_{T_2,K_2}(t,S(t))$, $T<\min(T_1,T_2)$,then the pricing of the complex chooser option is[1]

$$h(T,S(T))=\max(C_{T_1,K_1}(T,S(T)),P_{T_2,K_2}(T,S(T)). \tag{1}$$

The stock market is described by a probability space$(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$equipped with the natural σ- filtration of a standard motion $\{B_t,0\leqslant t\leqslant T\}$,indexed on the time$[0,T]$.

1.1 Preliminary knowledge

Lemma 1[2] Let the calll options with maturity time is $T_1\geqslant T$ and strike price is K_1,let the put options with maturity time is $T_2\geqslant T$ and the strike price is K_2. Then there exists a certain $S^*>0$,such that $C_{T_1,K_1}(T,S^*)=P_{T_2,K_2}(T,S^*)$.

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From Lemma 1,The payoff(1)of the complex chooser option with the chooser date T is given by

$$h(T,S(T))=C_{T_1,K_1}(T,S(T))I_{\{S(T)\geqslant S^*\}}+P_{T_2,K_2}(T,S(T))I_{\{S(T)<S^*\}}, \tag{2}$$

where the $I_{\{\cdot\}}$ is indicator function.

Theorem 1[3] Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion build on a probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$ and let $(\mathscr{F}_t)_{t\geqslant 0}$ be its natural filtration. Let $\{\lambda_t,t\geqslant 0\}$ be an adapted process,with respect to the filtration $\mathscr{F}_t$ such that $E^P[\exp\{\int_0^T\frac{1}{2}\lambda_s^2\mathrm{d}s\}]<\infty$,where $E^P(\cdot)$ is expectation under P. We denoted Rondon-Nikodym derivative

$$Z_t=\exp\{-\int_0^t\lambda_S\mathrm{d}B_s-\frac{1}{2}\int_0^t\lambda_S^2\mathrm{d}s\}<\infty,\quad \forall 0\leqslant t\leqslant T,$$

then Z_t is a continuous $\mathscr{F}_t$-Martingale. So we can construct a equivalent martingale measure $\tilde{P}$,such that $\tilde{P}(A)=E_P(I_AZ_T)$, $\forall A\in\mathscr{F}_t$,from Girsanov theorem $\tilde{B}_t=B_t+\int_0^t\lambda_s\mathrm{d}s,0\leqslant t\leqslant T$,is a standard $\mathscr{F}_t$-Brownian motion under $\tilde{P}$.

Let M is a continuous $\mathscr{F}_t$-local martingale,$(\sigma_t)_{t\geqslant 0}$ is $\mathscr{F}_t$ adapted process,satisfying

$$E^P[\int_0^T\sigma_s^2\mathrm{d}s]<\infty,\langle M\rangle(t)=\int_0^t\sigma_s^2\mathrm{d}s,\quad \forall 0\leqslant t\leqslant T.$$

So we have martingale representation theorem as follows.

Theorem 2[3] Let $\sigma_t\neq 0,P-a.s.\ \forall 0\leqslant t\leqslant T$,then there exists a standard $\mathscr{F}_t$-Brownian motion $(B_t)_{t\geqslant 0}$,then M such that $M(t)=M(0)+\int_0^t\sigma_S\mathrm{d}B_s$, $\forall 0\leqslant t\leqslant T$.

Theorem 3 Let $X\sim N(\mu_X,\sigma_X^2)$,$Y\sim N(\mu_Y,\sigma_Y^2)$,then

$$E(\mathrm{e}^XI_{\{X\geqslant a,Y\geqslant b\}})=\mathrm{e}^{\mu_X+\frac{\sigma_X^2}{2}}M\left(\frac{-a+\mu_X+\sigma_X^2}{\sigma_X},\frac{-b+\mu_Y+\mathrm{Cov}(X,Y)}{\sigma_Y},\rho\right),$$

where $\mathrm{Cov}(X,Y)$ is the covariance of X,Y,ρ is the related coefficient $X,Y,M(\cdot,\cdot,\cdot)$ is bivariate normal joint distribution function of a random variable.

$$M(x,y,\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^x\int_{-\infty}^y\exp\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\}\mathrm{d}u\mathrm{d}v.$$

1.2 The martingale pricing for complex chooser option

We consider a continuous time of financial market,given a probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$ equipped with a filtration,we assume a financial model consisting of two risky underlying assets,namely a bank account and a stock. We shall assume the stock $S(t)$ follows a continuous generalized exponential Ornstein-Uhlenbeck process

$$\mathrm{d}S(t)=S(t)[(\mu(t)-a\ln S(t))\mathrm{d}t+\sigma(t)\mathrm{d}B(t)],S(0)=S. \tag{3}$$

The bank $P(t)$ follows:

$$\mathrm{d}P(t)=P(t)r(t)\mathrm{d}t,P(T)=1, \tag{4}$$

where $S>0$, $((B_t)_{0\leqslant t\leqslant T})$ denote a Brownian Motions on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$. We suppose that the stock appreciation rate is $\mu(t)$ and the volatility is $\sigma(t)$. The market interest rate is $r(t)$. $\mu(t),\sigma(t),r(t)$ are satisfied as $\int_0^T\mu(t)\mathrm{d}t<\infty$, $\int_0^T\sigma^2(t)\mathrm{d}t<\infty$, $\int_0^Tr(t)\mathrm{d}t<\infty$.

Now,we give the martingale pricing formula for Complex chooser option.

Theorem 4[4] We assumed that the stock price follows a continuous generalized exponential O-U process model(3),let $\theta(t)=\frac{\mu(t)-r(t)-a\ln S(t)}{\sigma(t)}$ and such that $E^P[\exp\{\frac{1}{2}\int_0^T\theta^2(s)\mathrm{d}s\}]<\infty$,then the martingale pricing of the Complex chooser option at time $t(0\leqslant t\leqslant T)$ is

$$CCO(t)=S(t)M(a_1,b_1,\rho_1)-K_1\exp\{-\int_t^{T_1}r(s)\mathrm{d}s\}M(a_2,b_2,\rho_2)-S(t)M(-a_1,-b_3,\rho_2)+$$
$$K_2\exp\{-\int_t^{T_2}r(s)\mathrm{d}s\}M(-a_2,-b_4,\rho_2),$$

where

$$X=\int_t^T\sigma(s)\mathrm{d}W(s),Y=\int_t^{T_1}\sigma(s)\mathrm{d}W(s),Z=\int_t^{T_2}\sigma(s)\mathrm{d}W(s),$$

$$\sigma_X^2=\int_t^T\sigma^2(s)\mathrm{d}s,\sigma_Y^2=\int_t^{T_1}\sigma^2(s)\mathrm{d}s,\sigma_Z^2=\int_t^{T_2}\sigma^2(s)\mathrm{d}s,$$

$$a_1=\frac{1}{\sigma_X}(\ln\frac{S(t)}{S*}+\int_t^T r(s)\mathrm{d}s+\frac{1}{2}\sigma_X^2),a_2=\frac{1}{\sigma_X}(\ln\frac{S(t)}{S*}+\int_t^T r(s)\mathrm{d}s-\frac{1}{2}\sigma_X^2),$$

$$b_1=\frac{1}{\sigma_Y}(\ln\frac{S(t)}{K_1}+\int_t^{T_1} r(s)\mathrm{d}s+\frac{1}{2}\sigma_Y^2),b_2=\frac{1}{\sigma_Y}(\ln\frac{S(t)}{K_1}+\int_t^{T_1} r(s)\mathrm{d}s-\frac{1}{2}\sigma_Y^2),$$

$$b_3=\frac{1}{\sigma_Z}(\ln\frac{S(t)}{K_2}+\int_t^{T_2} r(s)\mathrm{d}s+\frac{1}{2}\sigma_Z^2),b_4=\frac{1}{\sigma_Z}(\ln\frac{S(t)}{K_2}+\int_t^{T_2} r(s)\mathrm{d}s-\frac{1}{2}\sigma_Z^2),$$

$$\rho_1=\frac{\sigma_X}{\sigma_Y},\rho_2=\frac{\sigma_X}{\sigma_Z},\quad W(t)=B(t)+\int_0^t\theta(s)\mathrm{d}s.$$

Proof　We assumed that the stock price follows a continuous generalized exponential O-U process model(3), let $\theta(t)=\frac{\mu(t)-r(t)-a\ln S(t)}{\sigma(t)}$, such that $E^P[\exp\{\frac{1}{2}\int_0^T\theta^2(s)\mathrm{d}s\}]<\infty$, there exists an equivalent martingale measure Q satisfying Radon-Nikodyn derivative $\frac{\mathrm{d}Q}{\mathrm{d}P}|F_T=\exp\{-\frac{1}{2}\int_0^T\theta^2(t)\mathrm{d}t-\int_0^T\theta(t)\mathrm{d}B(t)\}<\infty$, such that the discounted stock price process $S^*(t)$ on a probability measure Q is the martingale process, $S^*(t)=S(t)\exp\{-\int_0^t r(s)\mathrm{d}s\}$. Let $W(t)=B(t)+\int_0^t\theta(s)\mathrm{d}s$, according to Girsanov theorem[3], $W(t)$ denote an independent standard Brownian Motions on a probability measure Q, (3) can be written as

$$\mathrm{d}S(t)=S(t)(r(t)\mathrm{d}t+\sigma(t)\mathrm{d}W(t)).\tag{5}$$

We can obtain that the stock price at time t,T,T_1,T_2 are:

$$S(t)=S\exp\{\int_0^t(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s+\int_0^t\sigma(s)\mathrm{d}W(s)\}.\tag{6}$$

$$S(T)=S(t)\exp\{\int_t^T(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s+\int_t^T\sigma(s)\mathrm{d}W(s)\}.\tag{7}$$

$$S(T_1)=S(t)\exp\{\int_t^{T_1}(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s+\int_t^{T_1}\sigma(s)\mathrm{d}W(s)\}.\tag{8}$$

$$S(T_2)=S(t)\exp\{\int_t^{T_2}(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s+\int_t^{T_2}\sigma(s)\mathrm{d}W(s)\}.\tag{9}$$

From(2), then the martingale pricing of the complex chooser option on $t(0\leqslant t\leqslant T)$ on a probability measure Q is:

$$\begin{aligned}CCO(t)=E^Q\{\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}[C_{T_1,K_1}(T,S(T))I_{\{S(T)\geqslant S^*\}}+P_{T_2,K_2}I_{\{S(T)<S^*\}}]|\mathscr{F}_t\}=\\ E^Q\{\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\{E^Q[\mathrm{e}^{-\int_T^{T_1}r(s)\mathrm{d}s}(S(T_1)-K_1)I_{\{S(T_1)\geqslant K_1\}}I_{\{S(T)\geqslant S^*\}}+\\ \mathrm{e}^{-\int_T^{T_2}r(s)\mathrm{d}s}(K_2-S(T_2))I_{\{S(T_2)<K_2\}}I_{\{S(T)<S^*\}}]|\mathscr{F}_t\}|\mathscr{F}_t\}=\\ E^Q\{[\mathrm{e}^{-\int_t^{T_1}r(s)\mathrm{d}s}(S(T_1)-K_1)I_{\{S(T)\geqslant S^*,S(T_1)\geqslant K_1\}}+\\ \mathrm{e}^{-\int_T^{T_2}r(s)\mathrm{d}s}(K_2-S(T_2))I_{\{S(T)<S^*,S(T_2)<K_2\}}]|\mathscr{F}_t\}=\\ E^Q\{S(T_1)\mathrm{e}^{-\int_t^{T_1}r(s)\mathrm{d}s}I_{\{S(T)\geqslant S^*,S(T_1)\geqslant K_1\}}|\mathscr{F}_t\}-E^Q\{K_1\mathrm{e}^{-\int_t^{T_1}r(s)\mathrm{d}s}I_{\{S(T)\geqslant S^*,S(T_1)\geqslant K_1\}}|\mathscr{F}_t\}-\\ E^Q\{S(T_2)\mathrm{e}^{-\int_t^{T_2}r(s)\mathrm{d}s}I_{\{S(T)<S^*,S(T_2)<K_2\}}|\mathscr{F}_t\}+E^Q\{K_2\mathrm{e}^{-\int_t^{T_2}r(s)\mathrm{d}s}I_{\{S(T)<S^*,S(T_2)<K_2\}}|\mathscr{F}_t\}=\\ D_1-D_2-D_3+D_4.\end{aligned}$$

We denote that the joint distribution function of two dimensional standardized normal random variables is

$M(x,y,\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x}\int_{-\infty}^{y}\exp\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\}\mathrm{d}u\mathrm{d}v.$

Firstly we compute D_1、D_2, from (7),

$$S(T)\geqslant S^*\Leftrightarrow\int_t^T\sigma(s)\mathrm{d}W(s)\geqslant\ln\frac{S^*}{S(t)}-\int_t^T(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s,$$

We denote $C_1=\ln\frac{S^*}{S(t)}-\int_t^T(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s$, so $X\sim N(0,\sigma_X^2)$, $S(T)\geqslant S^*\Leftrightarrow X\geqslant C_1$, from (8) we have

$$S(T_1)\geqslant K_1\Leftrightarrow\int_t^{T_1}\sigma(s)\mathrm{d}W(s)\geqslant\ln\frac{K_1}{S(t)}-\int_t^{T_1}(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s,$$

note that $Y=\int_t^{T_1}\sigma(s)\mathrm{d}W(s)$, $\sigma_Y^2=\int_t^{T_1}\sigma^2(s)\mathrm{d}s$, $C_2=\ln\frac{K_1}{S(t)}-\int_t^{T_1}(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s$, then $Y\sim N(0,\sigma_Y^2)$ and $S(T_1)\geqslant K_1\Leftrightarrow Y\geqslant C_2$. Therefore

$$\mathrm{Cov}(X,Y)=E(XY)-EX\cdot EY=E(\int_t^{\mathrm{T}}\sigma(s)\mathrm{d}W(s)\cdot\int_t^{T_1}\sigma(s)\mathrm{d}W(s))=\sigma_X^2,$$

we obtain $\rho_1=\frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}=\frac{\sigma_X}{\sigma_Y}$, so

$$D_1=E^Q\{S(T_1)\,\mathrm{e}^{-\int_t^{T_1}r(s)\mathrm{d}s}I_{\{S(T)\geqslant S^*,S(T_1)\geqslant K_1\}}\mid F_t\}=S(t)M(a_1,b_1,\rho_1),$$

$$D_2=E^Q\{K_1\mathrm{e}^{-\int_t^{T_1}r(s)\mathrm{d}s}I_{\{S(T)\geqslant S^*,S(T_1)\geqslant K_1\}}\mid\mathscr{F}_t\}=K_1\mathrm{e}^{-\int_t^{T_1}r(s)\mathrm{d}s}M(a_2,b_2,\rho_1).$$

Now, we compute D_3、D_4, from (7), $S(T)<S^*\Leftrightarrow X<C_1$, from (9) we have

$$S(T_2)<K_2\Leftrightarrow\int_t^{T_2}\sigma(s)\mathrm{d}W(s)<\ln\frac{K_2}{S(t)}-\int_t^{T_2}(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s.$$

We denote $Z=\int_t^{T_2}\sigma(s)\mathrm{d}W(s)$, $\sigma_Z^2=\int_t^{T_2}\sigma^2(s)\mathrm{d}s$, $C_3=\ln\frac{K_2}{S(t)}-\int_t^{T_2}(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s$, then $Z\sim N(0,\sigma_Z^2)$ and $S(T_2)<K_2\Leftrightarrow Z<C_3$, So

$$\mathrm{Cov}(X,Z)=E(XZ)-EX\cdot EZ=E(\int_t^{\mathrm{T}}\sigma(s)\mathrm{d}W(s)\cdot\int_t^{T_2}\sigma(s)\mathrm{d}W(s))=\sigma_X^2,$$

$$D_3=E^Q\{S(T_2)\,\mathrm{e}^{-\int_t^{T_2}r(s)\mathrm{d}s}I_{\{S(T)<S^*,S(T_2)\geqslant K_2\}}\mid\mathscr{F}_t\}=S(t)M(-a_1,-b_3,\rho_2),$$

$$D_4=K_2\mathrm{e}^{-\int_t^{T_2}r(s)\mathrm{d}s}M(-a_2,-b_4,\rho_2),\rho_2=\frac{\mathrm{Cov}(X,Z)}{\sigma_X\sigma_Z}=\frac{\sigma_X}{\sigma_Z}.$$

From what we have been computed D_1,D_2,D_3,D_4, we can obtain the complex chooser option pricing at time $t(0\leqslant t\leqslant T)$ is $CCO(t)$.

2 Actuarial Methods for the Complex Chooser Options Pricing

Under the actuarial methods[5], the complex option can be executed for the conditions of the call option at time T is $\mathrm{e}^{-\int_t^T\beta(s)\mathrm{d}s}S(T)\geqslant\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}S^*$, be executed for the conditions of the put option at time T is $\mathrm{e}^{-\int_t^{\mathrm{T}}\beta(s)\mathrm{d}s}S(T)<\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}S^*$. So, the revenue for the complex chooser option is

$$h(T,S(T))=C_{T_1,K_1}(T,S(T))\,I_{\{\mathrm{e}^{-\int_t^T\beta(s)\mathrm{d}s}S(T)\geqslant\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}S^*\}}+P_{T_2,K_2}(T,S(T))\,I_{\{\mathrm{e}^{-\int_t^T\beta(s)\mathrm{d}s}S(T)<\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}S^*\}}.\quad(10)$$

Now, we give the actuarial methods for the complex chooser options pricing.

Definition 1 The expected interest rate $\int_0^t\beta(s)\mathrm{d}s$ of the stochastic process $\{S(t),t\geqslant0\}$ in the time interval $[0,t]$ is defined as $\exp\{\int_0^t\beta(s)\mathrm{d}s\}=\frac{ES(t)}{S(0)}$. $\beta(t)$ is continuously compounded interest rate of the $S(t)$ on time t, and $\int_0^t\beta(s)\mathrm{d}s<\infty$.

Theorem 5 We assume that stock price follows a continuous generalized exponential O-U process model (2), let $\theta(t)=\dfrac{\mu(t)-r(t)-a\ln S(t)}{\sigma(t)}$, satisfying $E^P\left[\exp\left\{\dfrac{1}{2}\int_0^T\theta^2(s)\mathrm{d}s\right\}\right]<\infty$, then the pricing of the complex chooser option at time $t(0\leqslant t\leqslant T)$ is

$$CCO'(t)=S(t)M(a_1',b_1',\rho_1')-K_1\exp\left\{-\int_t^{T_1}\beta(s)\mathrm{d}s-\int_T^{T_1}r(s)\mathrm{d}s\right\}M(a_2',b_2',\rho_1')-$$
$$S(t)M(-a_1',-b_3',\rho_2')+K_2\exp\left\{-\int_t^{T_2}\beta(s)\mathrm{d}s-\int_T^{T_2}r(s)\mathrm{d}s\right\}M(-a_2',-b_4',\rho_2'),$$

where,

$$X'=\mathrm{e}^{-aT}\int_t^T\mathrm{e}^{as}\sigma(s)\mathrm{d}B(s),Y'=\mathrm{e}^{-aT_1}\int_t^{T_1}\mathrm{e}^{as}\sigma(s)\mathrm{d}B(s),$$
$$Z'=\mathrm{e}^{-aT_2}\int_t^{T_2}\mathrm{e}^{as}\sigma(s)\mathrm{d}B(s),\sigma_{X'}^2=\mathrm{e}^{-2aT}\int_t^T\mathrm{e}^{2as}\sigma^2(s)\mathrm{d}s,$$
$$\sigma_{Y'}^2=\mathrm{e}^{-2aT_1}\int_t^{T_1}\mathrm{e}^{2as}\sigma^2(s)\mathrm{d}s,\sigma_{Z'}^2=\mathrm{e}^{-2aT_2}\int_t^{T_2}\mathrm{e}^{2as}\sigma^2(s)\mathrm{d}s,$$
$$a_1'=\frac{1}{\sigma_{X'}}\left[\ln\frac{S(t)}{S*}+\int_t^T r(s)\mathrm{d}s+\left(\mathrm{e}^{-a(T_1-T)}-\frac{1}{2}\right)\sigma_{X'}^2\right],$$
$$a_2'=\frac{1}{\sigma_{X'}}\left[\ln\frac{S(t)}{S*}+\int_t^T r(s)\mathrm{d}s-\frac{1}{2}\sigma_{X'}^2\right],$$
$$b_1'=\frac{1}{\sigma_{Y'}}\left[\ln\frac{S(t)}{K_1}+\int_t^{T_1}r(s)\mathrm{d}s+\int_t^{T_1}\beta(s)\mathrm{d}s+\frac{1}{2}\sigma_{Y'}^2\right],$$
$$b_2'=\frac{1}{\sigma_{Y'}}\left[\ln\frac{S(t)}{K_1}+\int_T^{T_1}r(s)\mathrm{d}s+\int_t^{T_1}\beta(s)\mathrm{d}s-\frac{1}{2}\sigma_{Y'}^2\right],$$
$$b_3'=\frac{1}{\sigma_{Z'}}\left[\ln\frac{S(t)}{K_2}+\int_T^{T_2}r(s)\mathrm{d}s+\int_t^{T_1}\beta(s)\mathrm{d}s+\frac{1}{2}\sigma_{Z'}^2\right],$$
$$b_4'=\frac{1}{\sigma_{Z'}}\left[\ln\frac{S(t)}{K_2}+\int_T^{T_2}r(s)\mathrm{d}s+\int_t^{T_1}\beta(s)\mathrm{d}s-\frac{1}{2}\sigma_{Z'}^2\right],$$
$$\int_t^{\mathrm{T}}\beta(s)\mathrm{d}s=[\mathrm{e}^{-a(T-t)}-1]\ln S(t)+\mathrm{e}^{-aT}\int_t^T\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\frac{1}{2}\sigma_{X'}^2,$$
$$\rho_1'=\mathrm{e}^{-a(T-t)}\frac{\sigma_{X'}}{\sigma_{Y'}},\rho_2'=\mathrm{e}^{-a(T_2-T)}\frac{\sigma_{X'}}{\sigma_{Z'}}.$$

Proof We assume that stock price follows a continuous generalized exponential O-U process model(2), by Itô formula, we can obtain that the stock price at time t is:

$$S(t)=S^{\mathrm{e}^{-at}}\exp\left\{\mathrm{e}^{-at}\int_0^t\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\mathrm{e}^{-at}\int_0^t\mathrm{e}^{as}\sigma(s)\mathrm{d}B(s)\right\},$$

so the stock price on chooser date T is

$$S(T)=S^{\mathrm{e}^{-a(T-t)}}(t)\exp\left\{\mathrm{e}^{-aT}\int_t^T\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\mathrm{e}^{-aT}\int_t^T\mathrm{e}^{as}\sigma(s)\mathrm{d}B(s)\right\},\tag{11}$$

$$ES(T)=S^{\mathrm{e}^{-a(T-t)}}(t)\exp\left\{\mathrm{e}^{-aT}\int_t^T\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\frac{1}{2}\mathrm{e}^{-2aT}\int_t^T\mathrm{e}^{2as}\sigma^2(s)\mathrm{d}s\right\}.\tag{12}$$

The expected return at the time index set $[t,T]$ is:

$$\int_t^T\beta(s)\mathrm{d}s=(\mathrm{e}^{-a(T-t)}-1)\ln S(t)+\mathrm{e}^{-aT}\int_t^T\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\frac{1}{2}\mathrm{e}^{-2aT}\int_t^T\mathrm{e}^{2as}\sigma^2(s)\mathrm{d}s.$$

The stock price at the maturity date T_1 of the call option is:

$$S(T_1)=S^{\mathrm{e}^{-a(T_1-t)}}(t)\exp\{\mathrm{e}^{-aT_1}\int_t^{T_1}\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\mathrm{e}^{-aT_1}\int_t^{T_1}\mathrm{e}^{as}\sigma(s)\mathrm{d}B(s)\},\tag{13}$$

$$ES(T_1)=S^{\mathrm{e}^{-a(T_1-t)}}(t)\exp\{\mathrm{e}^{-aT_1}\int_t^{T_1}\mathrm{e}^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]\mathrm{d}s+\frac{1}{2}\mathrm{e}^{-2aT_1}\int_t^{T_1}\mathrm{e}^{2as}\sigma^2(s)\mathrm{d}s\}.\tag{14}$$

The stock price at the maturity date T_2 of the put option is:

$$S(T_2)=S^{e^{-a(T_2-t)}}(t)\exp\{e^{-aT_2}\int_t^{T_2} e^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]ds+e^{-aT_2}\int_t^{T_2} e^{as}\sigma(s)dB(s)\},\tag{15}$$

$$ES(T_2)=S^{e^{-a(T_2-t)}}(t)\exp\{e^{-aT_2}\int_t^{T_2} e^{as}\left[\mu(s)-\frac{1}{2}\sigma^2(s)\right]ds+\frac{1}{2}e^{-2aT_2}\int_t^{T_2} e^{2as}\sigma^2(s)ds\}.\tag{16}$$

From(10), using conditional expectation of smoothness, insurance actuary pricing of the complex chooser option at time $t(0\leqslant t\leqslant T)$ is:

$$\begin{aligned}CCO'(t)=&E\{S(T_1)e^{-\int_t^{T_1}\beta(s)ds}I_{\{e^{-\int_t^T\beta(s)ds}S(T)\geqslant e^{-\int_t^T r(s)ds}S^*,e^{-\int_T^{T_1}\beta(s)ds}S(T_1)\geqslant e^{-\int_T^{T_1}r(s)ds}K_1\}}|\mathscr{F}_t\}-\\&E\{K_1\exp\{-\int_t^T\beta(s)ds-\int_T^{T_1}r(s)ds\}I_{\{e^{-\int_t^T\beta(s)ds}S(T)\geqslant e^{-\int_t^T r(s)ds}S^*,e^{-\int_T^{T_1}\beta(s)ds}S(T_1)\geqslant e^{-\int_T^{T_1}r(s)ds}K_1\}}|\mathscr{F}_t\}-\\&E\{S(T_2)e^{-\int_t^{T_2}\beta(s)ds}I_{\{e^{-\int_t^T\beta(s)ds}S(T)<e^{-\int_t^T r(s)ds}S^*,e^{-\int_T^{T_2}\beta(s)ds}S(T_2)<e^{-\int_T^{T_2}r(s)ds}K_2\}}|\mathscr{F}_t\}+\\&E\{K_2\exp\{-\int_t^T\beta(s)ds-\int_T^{T_2}r(s)ds\}I_{\{e^{-\int_t^T\beta(s)ds}S(T)<e^{-\int_t^T r(s)ds}S^*,e^{-\int_T^{T_2}\beta(s)ds}S(T_2)<e^{-\int_T^{T_2}r(s)ds}K_2\}}|\mathscr{F}_t\}=\\&D_1'-D_2'-D_3'+D_4'.\end{aligned}$$

Firstly, we obtain D_1'、D_2', from(11)and(12):

$$e^{-\int_t^T\beta(s)ds}S(T)\geqslant e^{-\int_t^T r(s)ds}S^*\Leftrightarrow e^{-aT}\int_t^T e^{as}\sigma(s)dB(s)\geqslant\frac{1}{2}e^{-2aT}\int_t^T e^{2as}\sigma^2(s)ds+\ln\frac{S^*}{S(t)}-\int_t^T r(s)ds.$$

We denote $C_1'=\frac{1}{2}e^{-2aT}\int_t^T e^{2as}\sigma^2(s)ds+\ln\frac{S^*}{S(t)}-\int_t^T r(s)ds$. Hence $X'\sim N(0,\sigma_{X'}^2)$, and $e^{-\int_t^T\beta(s)ds}S(T)\geqslant e^{-\int_t^T r(s)ds}S^*\Leftrightarrow X'\geqslant C_1'$. From(13)and(14)we have

$$e^{-\int_T^{T_1}\beta(s)ds}S(T_1)\geqslant e^{-\int_T^{T_1}r(s)ds}K_1\Leftrightarrow e^{-aT_1}\int_t^{T_1} e^{as}\sigma(s)dB(s)\geqslant\frac{1}{2}e^{-2aT_1}\int_t^{T_1} e^{2as}\sigma^2(s)ds+\ln\frac{K_1}{S(t)}-\int_T^{T_1} r(s)ds.$$

Note that $C_2'=\frac{1}{2}e^{-2aT_1}\int_t^{T_1}e^{2as}\sigma^2(s)ds+\ln\frac{K_1}{S(t)}-\int_t^{T_1}\beta(s)ds-\int_T^{T_1}r(s)ds$. Then $Y'\sim N(0,\sigma_{Y'}^2)$, and $e^{-\int_T^{T_1}\beta(s)ds}S(T_1)\geqslant e^{-\int_T^{T_1}r(s)ds}K_1\Leftrightarrow Y'\geqslant C_2'$. Therefore

$$\mathrm{Cov}(X',Y')=E(X'Y')-EX'\cdot EY'=e^{-a(T_1-T)}\sigma_{X'}^2,$$

we obtain $\rho_1'=\frac{\mathrm{Cov}(X',Y')}{\sigma_{X'}\sigma_{Y'}}=e^{-a(T_1-T)}\frac{\sigma_{X'}}{\sigma_{Y'}}$.

From Theorem 3 and(13),(14), we can see that

$$D_1'=S(t)M(a_1',b_1',\rho_1'),D_2'=K_1e^{\{-\int_t^T\beta(s)ds-\int_T^{T_1}r(s)ds\}}M(a_2',b_2',\rho_1'),$$

Then we compute D_3'、D_4', note that $e^{-\int_t^T\beta(s)ds}S(T)<e^{-\int_t^T r(s)ds}S^*\Leftrightarrow X'<C_1'$.

From(15),(16), we have:

$$e^{-\int_T^{T_2}\beta(s)ds}S(T_2)<e^{-\int_T^{T_2}r(s)ds}K_2\Leftrightarrow e^{-aT_2}\int_t^{T_2} e^{as}\sigma(s)dB(s)<\frac{1}{2}e^{-2aT_2}\int_t^{T_2} e^{2as}\sigma^2(s)ds+\ln\frac{K_2}{S(t)}-\int_t^T\beta(s)ds-\int_T^{T_1} r(s)ds.$$

We denote

$$C_3'=\frac{1}{2}e^{-2aT_2}\int_t^{T_2} e^{2as}\sigma^2(s)ds+\ln\frac{K_2}{S(t)}-\int_t^T\beta(s)ds-\int_T^{T_2} r(s)ds.$$

Then $Z'\sim N(0,\sigma_{Z'}^2)$, and $e^{-\int_T^{T_2}\beta(s)ds}S(T_2)<e^{-\int_T^{T_2}r(s)ds}K_2\Leftrightarrow Z'<C_3'$. So $\mathrm{Cov}(X',Z')=e^{-a(T_2-T)}\sigma_{X'}^2$, we obtain $\rho_2'=\frac{\mathrm{Cov}(X',Z')}{\sigma_{X'}\sigma_{Z'}}=e^{-a(T_2-T)}\frac{\sigma_{X'}}{\sigma_{Z'}}$.

From Theorem 3 and(15),(16), we have

$$D_3'=E\{S(T_2)\frac{S(t)}{ES(T_2)}I_{\{X'<C_1',Z'<C_3'\}}|\mathscr{F}_t\}=S(t)M(-a_1',-b_3',\rho_2'),$$

$$D_4' = K_2 e^{\{-\int_t^T \beta(s) ds - \int_T^{T_2} r(s) ds\}} M(-a_2', -b_4', \rho_2').$$

From what we have been computed D_1', D_2', D_3', D_4', we can obtain the complex chooser option pricing at time $t(0 \leqslant t \leqslant T)$ is $CCO'(t)$

[参考文献]

[1] NIU S M, XU Y. The pricing of European complex chooser option in fractional jump-diffusion process[J]. Mathematical theory and applications, 2012, 32(2): 39-46.

[2] KORN R, KORN E. Option pricing and portfolio optimization[M]. New York: American Mathematical Society, 2000.

[3] IOANIS K, STEVEN E S. Brownian motion and stochastic calculus[M]. New York: Publishing Corporation, 1990.

[4] STEIN E M, STEIN J C. Stock price distributions with stochastic volatility: an analytic approach[J]. The review of financial studies, 1991, 4: 727-752.

[5] BLADT M, RYDBERG H T. An actuarial approach to option pricing under the physical measure and without market assumption[J]. Insurance: mathematics and economic, 1998, 22(1): 65-73.

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