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具有 Radon 测度数据的单向流问题

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[摘要] 单相流问题可以从不可压流体力学中的一个简化的 Boussinesq 方程推导出来. Boussinesq 方程由不可压 Navier-Stokes 方程和一个非线性热传导方程耦合而成. 它在大气科学和海洋科学中有重要的应用. 本文的目的是要证明具有 Radon 测度数据的单相流问题至少存在一个整体的弱解. 我们利用正则化方法完成定理的证明. 首先,构造一系列逼近的正则化解;然后,利用方程的非线性微妙结构和一个推广的 Gronwall 不等式,建立良好的一致估计;最后,利用标准的 Aubin-Lions-Simon 紧致化原理、Lebesgue 控制收敛定理及非线性泛函分析中的一些结论,完成定理的证明. 本文的创新之处在于充分利用方程的细微非线性结构,获得精细的一致先验估计.

[关键词] Radon 测度,弱解,单相流

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A Unidirectional Flow Problem with Radon Measure Data

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Abstract:A unidirectional flow model is derived from a simplified Boussinesq system,which consists of a nonlinear heat equation coupled with the incompressible Navier-Stokes system. It has many important applications in atmosphere and ocean sciences. The aim of this paper is to prove the global existence of weak solutions to the unidirectional flow problem with Radon measure data. To achieve this,the regularized method is used. First,we construct the approximation strong solutions. Then,we apply a generalized Gronwall lemma to establish the uniform a priori estimates. Finally,we apply the standard compactness principle due to Aubin-Lions-Simon and thus the proof is finished. Here it should be note that the Gronwall inequality and the Lebesgue dominated convergence theorem are also used. The novelty of this paper may be lying in using the nonlinear subtle structure to obtain some fine uniform a priori estimates.

Key words:Radon measure,weak solutions,unidirectional flow

1994 年,Xu[1]研究了如下单向流问题:

$$u_t-\Delta \boldsymbol{u}=\sigma(u)\,|\,\nabla \boldsymbol{\varphi}|^2, \tag{1}$$

$$\varphi_t-\nabla\cdot(\sigma(u)\,\nabla \boldsymbol{\varphi})=G(t),\text{in } Q_T:=\Omega\times(0,T), \tag{2}$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}}=\varphi=0,\text{on } S_T:=\partial\Omega\times(0,T), \tag{3}$$

$$u(x,0)=u_0(x),\varphi(x,0)=\varphi_0(x)\text{ in } \Omega. \tag{4}$$

这里有界光滑区域 $\Omega\subseteq R^N(N\geqslant 1)$,$\boldsymbol{\nu}$ 是 $\partial\Omega$ 的单位外法向量,T 是任一给定正数. u 表示温度,φ 是流体速率. (1)是热传导方程,(2)是特殊情形下的 Navier-Stokes 方程. 因此这是一特殊情形的 Boussinesq 方程. 关于不可压 Navier-Stokes 方程和 Boussinesq 方程组的最新研究见参考文献[2-6].

Xu[1]证明当成立如下条件

$$G(t)\in L^2(0,T),u_0\in H^1(\Omega)\text{ 且 }\inf u_0>-\infty,\varphi_0\in H_0^1(\Omega)\cap L^\infty(\Omega), \tag{5}$$

$\sigma(u)$是连续函数且存在常数 σ_0、σ_1 使

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$$0<\sigma_0\leqslant\sigma\leqslant\sigma_1<+\infty. \tag{6}$$

则问题(1)~(4)至少存在一个弱解.

本文将证明如下定理:

定理 设(6)及如下条件成立

$$G(t)\in L^1(0,T),\varphi_0\in L^2(\Omega),u_0\in M_b(\Omega)(\text{有界 Radon 测度}), \tag{7}$$

则问题(1)~(4)至少存在一个弱解.

1 定理的证明

我们将采用正则化方法证明存在性定理. 设

$$G_\varepsilon(t)\in L^2(0,T),\varphi_{0\varepsilon}\in H_0^1(\Omega)\cap L^\infty(\Omega),u_{0\varepsilon}\in H^1(\Omega)\text{且}\inf u_{0\varepsilon}>-\infty, \tag{8}$$

且

$$\begin{aligned}&G_\varepsilon(t)\to G(t)\text{ in }L^1(0,T),\\&\varphi_{0\varepsilon}\to\varphi_0\text{ in }L^2(\Omega),\\&u_{0\varepsilon}\to u_0\text{ 在分布的意义下,且}\|u_{0\varepsilon}\|_{L^1(\Omega)}\leqslant\|u_0\|_{M_b(\Omega)}.\end{aligned} \tag{9}$$

设以 $G_\varepsilon(t),\varphi_{0\varepsilon},u_{0\varepsilon}$ 为已知数据的问题(1)~(4)的解为 $(u_\varepsilon,\varphi_\varepsilon)$[1],下面作解 $(u_\varepsilon,\varphi_\varepsilon)$ 与 ε 无关的一致估计,然后利用紧性定理完成定理的证明.

引理 1 设 $y(t)\in W_+^{1,1}(0,T)$,$g(t)$、$k(t)\in L_+^1(0,T)$,满足

$$\frac{\mathrm{d}y^2}{\mathrm{d}t}\leqslant 2gy^2+2ky,\quad y(0)\leqslant y_0, \tag{10}$$

则

$$y(t)\leqslant\left(y(0)+\int_0^T k(s)\mathrm{d}s\right)\exp\left(\int_0^T g(t)\mathrm{d}t\right). \tag{11}$$

引理 2

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\|\varphi_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\leqslant C, \tag{12}$$

$$\|\varphi_{\varepsilon t}\|_X\leqslant C,X=L^1(0,T)+L^2(0,T;H^{-1}(\Omega)). \tag{13}$$

这里及以后 C 均表示与 ε 无关的正常数.

证明 在方程(2)两端同时乘以 φ_ε 并积分,有

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\varphi_\varepsilon^2\mathrm{d}x+\int_\Omega\sigma(u_\varepsilon)|\nabla\boldsymbol{\varphi}_\varepsilon|^2\mathrm{d}x\leqslant|\Omega|^{1/2}|G_\varepsilon(t)|\,\|\varphi_\varepsilon\|_{L^2(\Omega)}. \tag{14}$$

利用引理 1 即得(12). 由(12)及方程(2)即得(13).

下面利用文献[7]中方法对 u_ε 给出先验估计.

引理 3

$$\|u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}\leqslant C. \tag{15}$$

证明 取 Lipschitz 连续函数 $\psi(s):\mathbf{R}\to[-1,1]$ 为:

$$\psi(s):=\begin{cases}1 & s>1,\\ s & -1\leqslant s\leqslant 1,\\ -1 & s<-1.\end{cases} \tag{16}$$

在方程(1)两端同时乘以 $\psi(u_\varepsilon)\chi_{(0,t)}$ 积分,有

$$\int_\Omega\Phi(u_\varepsilon)\mathrm{d}x+\int_0^t\int_\Omega\nabla\boldsymbol{u}_\varepsilon\cdot\nabla\boldsymbol{\psi}(u_\varepsilon)\mathrm{d}x\mathrm{d}t\leqslant\int_\Omega\Phi(u_{0\varepsilon})\mathrm{d}x+\int_0^t\int_\Omega\psi(u_\varepsilon)\sigma(u_\varepsilon)|\nabla\boldsymbol{\phi}_\varepsilon|^2\mathrm{d}x\mathrm{d}t.$$

从而

$$\int_\Omega|u_\varepsilon|\mathrm{d}x\leqslant\int_\Omega\Phi(u_{0\varepsilon})\mathrm{d}x+\int_0^t\int_\Omega\psi(u_\varepsilon)\sigma(u_\varepsilon)|\nabla\boldsymbol{\phi}_\varepsilon|^2\mathrm{d}x\mathrm{d}t\leqslant C. \tag{17}$$

这里

$$\Phi(s):=\int_0^s\psi(\sigma)\mathrm{d}\sigma,\ \int_\Omega\Phi(u_{0\varepsilon})\mathrm{d}x\leqslant\int_\Omega|u_{0\varepsilon}|\mathrm{d}x. \tag{18}$$

引理4

$$\| u_\varepsilon \|_{L^q(0,T;W^{1,q}(\Omega))} \leqslant C, \ \forall q < \frac{N+2}{N+1}, \tag{19}$$

$$\| u_{\varepsilon t} \|_X \leqslant C. \tag{20}$$

其中 $X := L^1(0,T;L^1(\Omega)) + L^1(0,T;(W^{1,q}(\Omega))^*)$.

证明　利用(19)及方程(1)而证(20)成立. 只须证明(19)成立,为此取 Lipschitz 连续函数 $\psi(s):\mathbf{R}\to[-1,1]$为

$$\psi(s) := \begin{cases} 1 & s>n+1, \\ s-n & n\leqslant s\leqslant n+1, \\ 0 & -n<s<n, \\ s+n & -n-1<s<-n, \\ -1 & s\leqslant -n-1. \end{cases} \tag{21}$$

在方程(1)两端同时乘以 $\psi(u_\varepsilon)$并积分,可得到

$$\int_\Omega\int_0^{u_\varepsilon} \psi(\sigma)\mathrm{d}\sigma\mathrm{d}x + \iint_{A_n} |\nabla \boldsymbol{u}_\varepsilon|^2 \mathrm{d}x\mathrm{d}t = \int_\Omega\int_0^{u_{0\varepsilon}} \psi(\sigma)\mathrm{d}\sigma\mathrm{d}x + \int_0^t\int_\Omega \psi(u_\varepsilon)\sigma(u_\varepsilon)|\nabla \boldsymbol{\phi}_\varepsilon|^2\mathrm{d}x\mathrm{d}t.$$

从而

$$\iint_{A_n} |\nabla \boldsymbol{u}_\varepsilon|^2 \mathrm{d}x\mathrm{d}t \leqslant \int_0^T\int_\Omega \psi(u_\varepsilon)\sigma(u_\varepsilon)|\nabla \boldsymbol{\phi}_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \int_\Omega |u_{0\varepsilon}|\mathrm{d}x \leqslant C, \tag{22}$$

其中 $A_n = \{(x,t)\in\Omega\times(0,T);n\leqslant|u_\varepsilon(x,t)|\leqslant n+1\}$.

取 $\gamma = \frac{N+1}{N}q$,则有

$$\iint_{A_\varepsilon} |\nabla \boldsymbol{u}_\varepsilon|^q \mathrm{d}x\mathrm{d}t \leqslant C(\mathrm{mes}\,A_n)^{1-\frac{q}{2}} \leqslant C\left(\iint_{A_n} |u_\varepsilon|^\gamma \mathrm{d}x\mathrm{d}t\right)^{1-\frac{q}{2}} \frac{1}{n^{\gamma(1-q/2)}},$$

从而

$$\int_0^T\int_\Omega |\nabla \boldsymbol{u}_\varepsilon|^q \mathrm{d}x\mathrm{d}t = \sum_{n=0}^{\infty}\iint_{A_n} |\nabla \boldsymbol{u}_\varepsilon|^q \mathrm{d}x\mathrm{d}t \leqslant C + C\left(\int_0^T\int_\Omega |u_\varepsilon|^\gamma \mathrm{d}x\mathrm{d}t\right)^{1-\frac{q}{2}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{\frac{N+1}{N}(2-q)}}\right)^{\frac{q}{2}}. \tag{23}$$

利用插值不等式

$$\| z \|_{L^\gamma(\Omega)} \leqslant \| z \|_{L^1(\Omega)}^{\theta} \| z \|_{L^{q^*}(\Omega)}^{1-\theta}, 1-\theta = \frac{1-\gamma}{1-q^*}\cdot\frac{q^*}{\gamma} = \frac{q}{\gamma}, q^* = \frac{Nq}{N-q}, \tag{24}$$

及 Sobolev-Poincaré 不等式

$$\left\| z - \frac{1}{|\Omega|}\int_\Omega z\mathrm{d}x \right\|_{L^{q^*}(\Omega)} \leqslant C \| \nabla \boldsymbol{z} \|_{L^q(\Omega)}. \tag{25}$$

可得到

$$\int_0^T\int_\Omega |u_\varepsilon|^\gamma \mathrm{d}x\mathrm{d}t \leqslant C\int_0^T\int_\Omega |\nabla \boldsymbol{u}_\varepsilon|^q \mathrm{d}x\mathrm{d}t + C. \tag{26}$$

将(26)代入(23)并利用(25)即得到(19).

利用 Simon J 紧性原理[8],可得

$$u_\varepsilon \to u \text{ strongly in } L^q(\Omega\times(0,T)) \text{ and a.e. } \Omega\times(0,T), \tag{27}$$

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^q(0,T;W^{1,q}(\Omega)), \tag{28}$$

$$\varphi_\varepsilon \to \varphi \text{ strongly in } L^2(\Omega\times(0,T)) \text{ and a.e. } \Omega\times(0,T), \tag{29}$$

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{30}$$

利用 σ 的连续性及有界性及(27)可知

$$\sigma(u_\varepsilon) \to \sigma(u) \text{ strongly in } L^p(\Omega\times(0,T)), \ \forall p>1, \tag{31}$$

由(30)、(31)可知

$$\sigma(u_\varepsilon)\nabla\boldsymbol{\varphi}_\varepsilon \to \sigma(u)\nabla\boldsymbol{\varphi} \text{ 在分布意义下} \tag{32}$$

由(9)、(29)、(32)可知 φ 满足

$$\varphi_t - \nabla \cdot (\sigma(u) \nabla \boldsymbol{\varphi}) = G(t), \tag{33}$$

由(30)、(3)知

$$\varphi|_{S_T} = 0. \tag{34}$$

由 Simon[8] 紧性定理知

$$\varphi|_{t=0} = \varphi_0(x) \in L^2(\Omega). \tag{35}$$

这样我们可以证明

引理 5

$$\varphi_\varepsilon \to \varphi \text{ strongly in } L^2(0,T;H_0^1(\Omega)). \tag{36}$$

证明

$$(\varphi_\varepsilon - \varphi)_t - \nabla \cdot ((\sigma(u_\varepsilon) \nabla \boldsymbol{\varphi}_\varepsilon) - (\sigma(u) \nabla \boldsymbol{\varphi})) = G_\varepsilon(t) - G(t), \tag{37}$$

$$(\varphi_\varepsilon - \varphi)|_{S_T} = 0, \tag{38}$$

$$(\varphi_\varepsilon - \varphi)|_{t=0} = \varphi_{0\varepsilon} - \varphi_0 \to 0 \text{ strongly in } L^2(\Omega). \tag{39}$$

在方程(37)两端同时乘以 $\varphi_\varepsilon - \varphi$ 并积分,有

$$\frac{1}{2}\int_\Omega (\varphi_\varepsilon - \varphi)^2 \mathrm{d}x + \int_0^T\int_\Omega \sigma(u_\varepsilon) |\nabla(\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi})|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega (\sigma(u_\varepsilon) - \sigma(u)) \nabla \boldsymbol{\varphi} \cdot \nabla(\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi}) \mathrm{d}x\mathrm{d}t =$$

$$\frac{1}{2}\int_\Omega (\varphi_{0\varepsilon} - \varphi_0)^2 \mathrm{d}x + \int_0^T\int_\Omega (G_\varepsilon(t) - G(t))(\varphi_\varepsilon - \varphi) \mathrm{d}x\mathrm{d}t.$$

从而

$$\frac{1}{2}\int_\Omega (\varphi_\varepsilon - \varphi)^2 \mathrm{d}x + \frac{1}{2}\int_0^T\int_\Omega \sigma(u_\varepsilon) |\nabla(\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi})|^2 \mathrm{d}x\mathrm{d}t \leqslant \frac{1}{2}\int_\Omega (\varphi_{0\varepsilon} - \varphi_0)^2 \mathrm{d}x +$$

$$\frac{1}{2}\int_0^T\int_\Omega |\sigma(u_\varepsilon) - \sigma(u)|^2 |\nabla \boldsymbol{\varphi}|^2 \mathrm{d}x\mathrm{d}t +$$

$$\| \varphi_\varepsilon - \varphi \|_{L^\infty(0,T;L^2(\Omega))} |\Omega|^{1/2} \| G_\varepsilon(t) - G(t) \|_{L^1(0,T)}. \tag{40}$$

利用 Lebesgue 控制收敛定理可知

$$\int_0^T\int_\Omega |\sigma(u_\varepsilon - \sigma(u)|^2 |\nabla \boldsymbol{\varphi}|^2 \mathrm{d}x\mathrm{d}t \to 0 \text{ 当 } \varepsilon \to 0 \text{ 时}, \tag{41}$$

把(41)代入(40)即可知引理 5 成立.

定理的证明 利用引理 4、5 易证定理成立.

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