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具分布时滞和阻尼项的三阶中立型微分方程的振动性

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[**摘要**] 研究一类具分布时滞和阻尼项的三阶非线性中立项微分方程的振动性,利用广义 Riccati 变换技术及一些分析技巧,建立了该类方程的新的振动准则,通过实例加以验证.

[**关键词**] 振动性,分布时滞,微分方程,阻尼项

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Oscillation of Third-Order Neutral Differential Equations with Distributed Delays and Damping

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Abstract: The present paper focuses on the oscillation of the third-order nonliear neutral differential equations with distributed delays and damping. By applying the generalized Riccati transformation and some analytic techniques, we establish several oscillation criteria for the discussed equation, which show that any solution either oscillates or converges to zero. Finally, we gives some examples to prove the efficiency.

Key words: oscillation, distributed delays, differential equations, damping terms

本文考虑一类具阻尼项的三阶广义 Emden-Fowler 型分布时滞微分方程

$$[r(t)\psi(x(t))|z''(t)|^{\beta-1}z''(t)]'+f(t)\psi(x(t))|z''(t)|^{\beta-1}z''(t)+\int_c^d q(t,\xi)|x(\delta(t,\xi))|^{\gamma-1}x(\delta(t,\xi))\mathrm{d}\xi=0,\quad t\geqslant t_0>0 \tag{1}$$

振动性,其中 $z(t)=x(t)+\int_a^b p(t,\mu)x^{\alpha}(\tau(t,\mu))\mathrm{d}\mu$,并假设下列条件成立:

(H_1) $0<\alpha\leqslant 1,\beta>0,\gamma>0,\alpha,\beta,\gamma$ 均为奇正整数之商;

(H_2) $r(t),f(t)\in C^1([t_0,\infty),(0,\infty)),r'(t)\geqslant 0,q(t,\xi)\in C([t_0,\infty)\times[c,d],(0,\infty))$;

(H_3) $p(t,\mu)\in C([t_0,\infty)\times[a,b],R),0\leqslant p(t)\equiv\int_a^b p(t,\mu)\mathrm{d}\mu\leqslant P<1,\lim\limits_{t\to\infty}p(t)=0$;

(H_4) $\tau(t,\mu)\in C([t_0,\infty)\times[a,b],R),\dfrac{\partial\tau(t,\mu)}{\partial\mu}\geqslant 0,\tau(t,\mu)\leqslant t,\lim\limits_{t\to\infty}\min\limits_{\mu\in[a,b]}\tau(t,\mu)=+\infty$;

(H_5) $\delta(t,\xi)\in C([t_0,\infty)\times[c,d],R),\dfrac{\partial\delta(t,\xi)}{\partial\xi}\geqslant 0,\delta(t,\xi)\leqslant t,\lim\limits_{t\to\infty}\min\limits_{\xi\in[c,d]}\delta(t,\xi)=+\infty$.

(H_6) $\int_{t_0}^{\infty}\left(\dfrac{1}{E(t)r(t)}\right)^{\frac{1}{\beta}}\mathrm{d}t=\infty$,其中 $E(t)=\exp\left(\int_{t_0}^{t}\dfrac{f(s)}{r(s)}\mathrm{d}s\right)$.

(H_7) $\psi\in C^1(R,R)$,当 $x\neq 0$ 时,$\psi(x)>0$,且存在正常数 L,使得 $\psi(x)\leqslant L^{-1}$;

泛函微分方程的振动理论在控制工程、通信工程、机械工程、生物医学和力学等领域有着广泛的应

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用[1]. 近年来,三阶微分方程的振动性理论受到很大的关注[2-7]. 文[8]研究了方程(1)当$\psi(x(t))=1$, $f(t)=0$时的特殊情形的振动性. 本文的目的是继续文[8]的研究,将利用广义 Riccati 变换以及各种不等式技巧,将文[8]的结论推广到具阻尼项的更广泛的三阶微分方程.

1 引理

下面出现的函数不等式均假设对一切充分大的t成立.在定理的证明中,我们仅处理方程(1)的最终正解,最终负解处理类似.

引理 1　设$x(t)$是方程(1)的最终正解,则$z(t)$只有下列两种可能:

(1)$z(t)>0, z'(t)>0, z''(t)>0$;

(2)$z(t)>0, z'(t)<0, z''(t)>0$.

证明　证明过程与文[7]中的引理 4 证明类似,故省略.

引理 2[9]　设$0<\lambda\leqslant 1$,则

(1)$X^\lambda+Y^\lambda\leqslant 2^{1-\lambda}(X+Y)^\lambda$, X,Y为非负实数.

(2)$(1+X)^\lambda\leqslant 1+\lambda X$,其中$1+X>0$.

引理 3[9]　若存在$\theta>0, A>0, B>0$,则有　$Ax-Bx^{\frac{\theta+1}{\theta}}\leqslant\dfrac{\theta^\theta}{(\theta+1)^{\theta+1}}\dfrac{A^{\theta+1}}{B^\theta}$.

引理 4　设$x(t)$是方程(1)的最终正解,$z(t)$满足引理 1 情形(1),则

$$[r(t)\psi(x(t))(z''(t))^\beta]'+f(t)\psi(x(t))(z''(t))^\beta+Q_1(t)z^\gamma(\delta(t,c))\leqslant 0, \tag{2}$$

式中,

$$Q_1(t)=\left\{1-\left[\alpha 2^{1-\alpha}+\frac{(2^{1-\alpha}-1)}{k}\right]P\right\}^\gamma\int_c^d q(t,\xi)\mathrm{d}\xi. \tag{3}$$

证明　由$z(t)$的定义,条件(1)、(H_3)、(H_4)及引理 2 可得

$$x(t)\geqslant z(t)-2^{1-\alpha}(1+\alpha z(t))P+P=(1-\alpha 2^{1-\alpha}P)z(t)-(2^{1-\alpha}-1)P.$$

由条件(1)知,$z(t)>0, z'(t)>0$,于是$z(\delta(t,c))\geqslant z(\delta(t_1,c))=k>0, t\geqslant t_1$.因而

$$\int_c^d q(t,\xi)x^\gamma(\delta(t,\xi))\mathrm{d}\xi\geqslant z^\gamma(\delta(t,c))\left\{1-\left[\alpha 2^{1-\alpha}+\frac{(2^{1-\alpha}-1)}{k}\right]P\right\}^\gamma\int_c^d q(t,\xi)\mathrm{d}\xi,$$

从而式(2)成立.

引理 5[10]　设$z(t)>0, z'(t)>0, z''(t)>0, z'''(t)<0, t\geqslant t_0$,则存在$\eta\in(0,1)$和$t_\eta>t_0$使得

$$\frac{z(t)}{z'(t)}\geqslant\eta t,\quad t\geqslant t_\eta.$$

引理 6　设$x(t)$是方程(1)的最终正解,$z(t)$满足引理 1 情形(1),函数 若存在$\rho(t)\in C^1([t_0,\infty),(0,\infty))$,令广义 Riccati 变换

$$w(t)=\rho(t)\frac{r(t)\psi(x(t))(z''(t))^\beta}{(z'(\delta(t,c)))^\gamma}, \tag{4}$$

则有

$$w'(t)\leqslant-\rho(t)Q_1(t)(\eta\cdot\delta(t,c))^\gamma+Q_2(t)w(t)-\frac{\lambda L_0\delta'(t,c)m}{(\rho(t)r(t))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(t),\quad t>T. \tag{5}$$

式中,$Q_2(t)=\left|\dfrac{\rho'_+(t)}{\rho(t)}-\dfrac{f(t)}{r(t)}\right|$, $\rho'_+(t)=\max\{\rho'(t),0\}$, $T=\max\{T,T_2\}$, $\lambda=\max\{\beta,\gamma\}$, $L_0=\min\{L^{\frac{1}{\beta}},L^{\frac{1}{\gamma}}\}$,

$$m=\begin{cases}m_1, & \gamma>\beta>0,\\ m_2, & \beta\geqslant\gamma>0.\end{cases}$$

证明　由$w(t)$的定义及引理 4、引理 5 可得

$$w'(t)\leqslant-\rho(t)Q_1(t)(\eta\cdot\delta(t,c))^\gamma+Q_2(t)w(t)-\gamma\rho(t)\frac{r(t)(z''(t))^\beta z''(\delta(t,c))\delta'(t,c)}{(z'(\delta(t,c)))^{\gamma+1}}. \tag{6}$$

(1)当 $\gamma>\beta>0$ 时,由方程(1)知,$[r(t)E(t)\psi(x(t))(z''(t))^{\beta}]'\leqslant 0$,故当 t 充分大时,$z'''(t)\leqslant 0$,从而 $\dfrac{z''(\delta(t,c))}{z''(t)}\geqslant 1$.又因为当 $\gamma>\beta>0$ 时,$(z'(\delta(t,c))^{\frac{\gamma}{\beta}-1}$ 单调递增,故存在充分大的 $T_1>t_2$,使

$$(z'(\delta(t,c)))^{\frac{\gamma}{\beta}-1}\geqslant(z'(\delta(T_1,c)))^{\frac{\gamma}{\beta}-1},t>T_1,$$

记 $m_1=\min\{1,(z'(\delta(T_1,c))^{\frac{\gamma}{\beta}-1}\}$,则 $(z'(\delta(t,c))^{\frac{\gamma}{\beta}-1}\geqslant(z'(\delta(T_1,c))^{\frac{\gamma}{\beta}-1}\geqslant m_1,t>T_1$,从而(6)式即为

$$w'(t)\leqslant-\rho(t)Q_1(t)(\eta\cdot\delta(t,c))^{\gamma}+Q_2(t)w(t)-\frac{\beta L^{\frac{1}{\beta}}\delta'(t,c)m_1}{(\rho(t)r(t))^{\frac{1}{\beta}}}w^{\frac{\beta+1}{\beta}}(t).$$

(2)当 $\beta\geqslant\gamma>0$ 时,

由 $z'''(t)\leqslant 0$ 知,$z''(t)$ 单调递减,故存在充分大的 $T_2>t_2$,使得

$$\frac{z''(\delta(t,c))}{z''(t)}\geqslant 1,\frac{z''(t)}{(z''(t))^{\frac{\beta}{\gamma}}}=(z''(t))^{1-\frac{\beta}{\gamma}}\geqslant(z''(T_2))^{1-\frac{\beta}{\gamma}},t>T_2,$$

记 $m_2=\min\{1,(z''(T_2))^{1-\frac{\beta}{\gamma}}\}$,则 $\dfrac{z''(\delta(t,c))}{(z''(t))^{\frac{\beta}{\gamma}}}=\dfrac{z''(\delta(t,c))}{z''(t)}\times\dfrac{z''(t)}{(z''(t))^{\frac{\beta}{\gamma}}}\geqslant m_2$,从而(6)式即为

$$w'(t)\leqslant-\rho(t)Q_1(t)(\eta\cdot\delta(t,c))^{\gamma}+Q_2(t)w(t)-\frac{\gamma L^{\frac{1}{\gamma}}\delta'(t,c)m_2}{(\rho(t)r(t))^{\frac{1}{\gamma}}}w^{\frac{\gamma+1}{\gamma}}(t).$$

综合(1)和(2)可得,当 $\beta>0,\gamma>0$ 时,不等式(5)成立.

引理 7 设 $x(t)$ 是方程(1)的最终正解,$z(t)$ 满足引理 1 情形(2),若

$$\int_T^{+\infty}\int_v^{+\infty}\left[\frac{1}{r(u)}\int_u^{+\infty}\int_c^d q(s,\xi)\,\mathrm{d}\xi\mathrm{d}s\right]^{\frac{1}{\beta}}\mathrm{d}u\mathrm{d}v=+\infty,\tag{7}$$

则 $\lim\limits_{t\to\infty}x(t)=0$.

证明 设 $x(t)$ 是方程(1)的最终正解,$z(t)$ 满足引理 1 情形(2). 因为 $z(t)>0,z'(t)<0$,则由单调有界原理可知 $\lim\limits_{t\to\infty}z(t)$ 存在,记 $\lim\limits_{t\to\infty}z(t)=l$,则 $l\geqslant 0$.假设 $l>0$,$\lim\limits_{t\to\infty}z(t)=l$,$\lim\limits_{t\to\infty}p(t)=0$. 则对任意 ε,$0<\varepsilon<\min\{l,2^{-\alpha}l^{1-\alpha}\}$,可得 $l<z(t)<l+\varepsilon$,$0\leqslant p(t)<\varepsilon$,从而

$$x(t)=z(t)-\int_a^b p(t,\mu)x^{\alpha}(\tau(t,\mu))\,\mathrm{d}\mu\geqslant l-(l+\varepsilon)^{\alpha}\geqslant N(l+\varepsilon)>Nz(t),$$

式中,$N=\dfrac{l-(2l)^{\alpha}\varepsilon}{l+\varepsilon}>0$,于是

$$[r(t)E(t)\psi(x(t))(z''(t))^{\beta}]'\leqslant-E(t)\int_c^d q(t,\xi)[Nz(\delta(t,\xi))]^{\gamma}\mathrm{d}\xi\leqslant-E(t)(Nl)^{\gamma}\int_c^d q(t,\xi)\,\mathrm{d}\xi,$$

两边从 t 到 ∞ 积分,得到

$$r(t)E(t)(z''(t))^{\beta}\leqslant-(Nl)^{\gamma}L\int_t^{\infty}E(s)\int_c^d q(s,\xi)\,\mathrm{d}\xi\mathrm{d}s.$$

注意到 $E'(t)\geqslant 0$,于是有

$$z''(t)\geqslant(Nl)^{\frac{\gamma}{\beta}}L\left[r^{-1}(t)\int_t^{\infty}\int_c^d q(s,\xi)\,\mathrm{d}\xi\mathrm{d}s\right]^{\frac{1}{\beta}},$$

两边关于 u 从 v 到 $+\infty$ 积分,关于 v 从 T 到 $+\infty$ 积分得

$$z(T)\geqslant(Nl)^{\frac{\gamma}{\beta}}L\int_T^{+\infty}\int_v^{+\infty}\left[r^{-1}(t)\int_t^{\infty}\int_c^d q(s,\xi)\,\mathrm{d}\xi\mathrm{d}s\right]^{\frac{1}{\beta}}\mathrm{d}u\mathrm{d}v,$$

这与式(7)矛盾,因此 $\lim\limits_{t\to\infty}x(t)=\lim\limits_{t\to\infty}z(t)=0$.

2 主要结果与实例

定理 1 设式(7)成立,若

$$\lim_{t\to+\infty}\sup\int_T^t\left[\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}-\left(\frac{Q_2(s)}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{L_0m^{\lambda}(\delta'(s,c))^{\lambda}}\right]\mathrm{d}s=+\infty,\tag{8}$$

则方程(1)的任意解振动或收敛于零.

证明　设 $x(t)$ 为方程(1)的最终正解.当 $z(t)$ 满足引理 1 情形(1)时,由引理 6 得式(5)成立,再由引理 3 得

$$w'(t)\leqslant -\rho(t)Q_1(t)(\eta\delta(t,c))^{\gamma}+\left(\frac{Q_2(t)}{\lambda+1}\right)^{\lambda+1}\frac{\rho(t)r(t)}{L_0(m\delta'(t,c))^{\lambda}},$$

两边从 T 到 t 积分得

$$w(t)\leqslant w(T)-\int_T^t\left[\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}+\left(\frac{Q_2(s)}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{L_0(m\delta'(s,c))^{\lambda}}\right]\mathrm{d}s,$$

令 $t\to+\infty$,由式(8)可得 $w(t)\to-\infty$,这与 $w(t)>0$ 矛盾,从而 $x(t)$ 为方程(1)的一个振动解.

当 $z(t)$ 满足引理 1 情形(2)时,由式(7)和引理 7 知,$\lim\limits_{t\to\infty}x(t)=0$. 定理 1 得证.

令 $D=\{(t,s)\mid t\geqslant s\geqslant t_0\}$,$D_0=\{(t,s)\mid t>s\geqslant t_0\}$,称函数 $H(t,s)\in C(D,R)$ 属于ゐ类,记作 $H\in$ ゐ,如果 $H(t,t)=0,t\geqslant t_0$;$H(t,s)>0,(t,s)\in D_0$;$\dfrac{\partial H(t,s)}{\partial s}\leqslant 0,(t,s)\in D$;存在 $h\in C(D_0,R)$ 和 $\rho\in C^1([t_0,\infty),(0,\infty))$,使得 $\dfrac{\partial H(t,s)}{\partial s}+Q_2(s)H(t,s)=-h(t,s)H^{\frac{\lambda}{\lambda+1}}(t,s)$,$h\in C(D,R)$,$(t,s)\in D_0$.

定理 2　设式(7)成立,若

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}-\left(\frac{|h(t,s)|}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{L_0m^{\lambda}(\delta'(s,c))^{\lambda}}\right]\mathrm{d}s=+\infty,\tag{9}$$

则方程(1) 的任意解振动或收敛于零.

证明　设方程(1) 有非振动解 $x(t)$.不失一般性,设 $x(t)$ 最终为正,由引理 1,$z(t)$ 只可能有(1) 和(2) 两种情形.

首先,设 $z(t)$ 满足情形(1),由引理 6 得式(6) 成立,两边同乘以 $H(t,s)$,且两边从 T 到 t 积分得

$$\int_T^t H(t,s)\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}\mathrm{d}s\leqslant H(t,T)w(T)+\int_T^t\left[\left(\frac{h(t,s)}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{(m\delta'(s,c))^{\lambda}}\right]\mathrm{d}s.\tag{10}$$

整理得

$$w(t)\geqslant\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}-\left(\frac{|h(t,s)|}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{m^{\lambda}(\delta'(s,c))^{\lambda}}\right]\mathrm{d}s,\tag{11}$$

与式(9) 矛盾,从而 $x(t)$ 为方程(1) 的振动解.

其次,设 $z(t)$ 满足情形(2),由于式(7) 成立,由引理 7 知 $\lim\limits_{t\to\infty}x(t)=0$.故得证.

取 $H(t,s)=(t-s)^n$,则定理 2 可简化为 Kamenev 型振动结果:

推论 1　假设式(7) 成立,若

$$\lim_{t\to+\infty}\sup\frac{1}{H(t,T)}\int_T^t\left[(t-s)^n\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}-\left(\frac{|h(t,s)|}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{L_0m^{\lambda}(\delta'(s,c))^{\lambda}}\right]\mathrm{d}s=+\infty,$$

则方程(1)的任意解振动或收敛于零.

定理 3　假设式(7)成立,若

$$0<\inf_{s\geqslant T}\left[\lim_{t\to+\infty}\inf\frac{H(t,s)}{H(t,T)}\right]\leqslant+\infty,\tag{12}$$

$$\lim_{t\to+\infty}\sup\frac{1}{H(t,T)}\int_T^t\frac{\rho(s)r(s)|h(t,s)|^{\lambda+1}}{L_0(\lambda m\delta'(s,c))^{\lambda}}\mathrm{d}s<+\infty,\tag{13}$$

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}-\left(\frac{|h(t,s)|}{\lambda+1}\right)^{\lambda+1}\frac{\rho(s)r(s)}{m^{\lambda}L_0(\delta'(s,c))^{\lambda}}\right]\mathrm{d}s\geqslant A(T),\tag{14}$$

$$\lim_{t\to+\infty}\sup\int_T^t\frac{\lambda L_0m\delta'(s,c)}{(\rho(s)r(s))^{\frac{1}{\lambda}}}A_+^{\frac{\lambda+1}{\lambda}}(s)\mathrm{d}s=+\infty,\tag{15}$$

式中,$A_+(s)=\max\{A(s),0\}$,则方程(1)的任意解振动或收敛于零.

证明 设方程(1)有非振动解 $x(t)$.不失一般性,设 $x(t)$ 最终为正.

首先,设 $z(t)$ 满足引理 1 中情形(1),由定理 2 的证明可知式(11)成立,再由(15)可得

$$w(T)\geqslant A(T). \tag{16}$$

利用式(10)和(15)可得

$$\liminf_{t\to+\infty}\left(\frac{1}{H(t,T)}\int_T^t H(t,s)\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s-\frac{1}{H(t,T)}\int_T^t|h(t,s)|w(s)H^{\frac{\lambda}{\lambda+1}}(t,s)\,\mathrm{d}s\right)\leqslant$$

$$w(T)-\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)Q_1(s)(\eta\delta(s,c))^{\gamma}\mathrm{d}s\leqslant w(T)-A(T)<+\infty. \tag{17}$$

令

$$F(t)=\frac{1}{H(t,T)}\int_T^t H(t,s)\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s,\ G(t)=\frac{1}{H(t,T)}\int_T^t|h(t,s)|w(s)H^{\frac{\lambda}{\lambda+1}}(t,s)\,\mathrm{d}s,$$

由式(17)可得

$$\liminf_{t\to+\infty}[F(t)-G(t)]<+\infty. \tag{18}$$

情况 1 假设

$$\int_T^t\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s<+\infty \tag{19}$$

成立.则由式(16) 得

$$\int_T^t\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}A_+^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s\leqslant\int_T^t\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s<+\infty,$$

这与式(15) 矛盾,所以式(19) 不成立.

情况 2 假设

$$\int_T^t\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s=+\infty \tag{20}$$

成立.设 η 是一个充分小的正数,利用条件(12) 得

$$0<\eta<\inf_{s\geqslant T}\left[\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,T)}\right]\leqslant+\infty, \tag{21}$$

由式(20)可得,$\forall\mu>0$,有

$$\int_T^t\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda+1}{\lambda}}(s)\,\mathrm{d}s\geqslant\frac{\mu}{\eta},t>T, \tag{22}$$

利用条件(12)及(21)、(22),取 $T'>T$,利用分部积分公式有

$$F(t)\geqslant\frac{1}{H(t,T)}\int_{T'}^t\left(\frac{-\partial H(t,s)}{\partial s}\right)\frac{\mu}{\eta}\mathrm{d}s\geqslant\frac{H(t,T')}{H(t,T)}\frac{\mu}{\eta}\geqslant\mu. \tag{23}$$

由 μ 的任意性,以及式(18)和(22)有

$$\lim_{t\to+\infty}F(t)=\lim_{t\to+\infty}G(t)=+\infty. \tag{24}$$

取数列 $\{t_n\}_{n=1}^{\infty},t_n\to+\infty$,由式(18)和(24),存在 $M>0$,使得 n 充分大时

$$F(t_n)-G(t_n)\leqslant M,\quad G(t_n)-F(t_n)\geqslant-M.$$

故当 n 充分大时,有

$$\frac{G(t_n)}{F(t_n)}>\frac{1}{4},\quad\left[\frac{G(t_n)}{F(t_n)}\right]^{\lambda}>\left[\frac{1}{4}\right]^{\lambda},\frac{G^{\lambda+1}(t_n)}{F^{\lambda}(t_n)}>\frac{G(t_n)}{4^{\lambda}}\to+\infty. \tag{25}$$

利用 Schwarz 不等式得

$$G^{\frac{\lambda+1}{\lambda}}(t_n)=\left[\frac{1}{H(t_n,T)}\int_T^{t_n}\frac{|h(t_n,s)|^{\lambda+1}(\rho(s)r(s))}{\lambda L_0\delta'(s,c)m)^{\lambda}}\mathrm{d}s\right]^{\frac{1}{\lambda}}\cdot\left[\frac{1}{H(t_n,T)}\int_T^{t_n}\frac{\lambda L_0\delta'(s,c)m}{(\rho(s)r(s))^{\frac{1}{\lambda}}}w^{\frac{\lambda}{\lambda+1}}(s)H(t_n,s)\,\mathrm{d}s\right]$$

两边同时除以 $F(t_n)$,由式(15)得

$$\frac{G^{\frac{\lambda+1}{\lambda}}(t_n)}{F(t_n)}=\frac{1}{H(t_n,T)}\int_T^{t_n}\frac{\rho(s)r(s)|h(t_n,s)|^{\lambda+1}}{L_0(\lambda\delta'(s,c)m)^{\lambda}}\mathrm{d}s<+\infty\ ,$$

这与不等式(25)矛盾,所以假设(20)不成立.

综合情况1与情况2的证明,因为式(19)和(20)都不成立,所以原假设不成立,从而 $x(t)$ 为方程(1)的振动解.

其次,设 $z(t)$ 满足情形(2),由于式(7)成立,由引理7知$\lim\limits_{t\to\infty}x(t)=0$.故得证.

例1　考虑下列三阶阻尼中立型微分方程

$$\left[t^2\frac{1+x^2(t)}{2+x^2(t)}\left(\left(x(t)+\int_0^1\frac{4\mu}{3t^2}x^{\frac{1}{3}}\left(\frac{t+\mu}{3}\right)\mathrm{d}\mu\right)''\right)^5\right]'+t\frac{1+x^2(t)}{2+x^2(t)}\left((x(t)+\int_0^1\frac{4\mu}{3t^2}x^{\frac{1}{3}}\left(\frac{t+\mu}{3}\right)\mathrm{d}\mu)''\right)^5+$$
$$\int_0^1\frac{64\xi}{t^3}x^7\left(\frac{t+\xi}{2}\right)\mathrm{d}\xi=0,t\geqslant 1. \tag{26}$$

则 $\psi(x(t))=\dfrac{1+x^2(t)}{2+x^2(t)}\leqslant 1=L,\lambda=\min\{\beta,\gamma\}=5$ 且

$$\int_{t_0}^{\infty}\left(\frac{1}{E(t)r(t)}\right)^{\frac{1}{\beta}}\mathrm{d}t=\int_{t_0}^{\infty}(t^{-3})^{\frac{1}{5}}\mathrm{d}t=\infty\ ,p(t)=\int_0^1\frac{4\mu}{3t^2}\mathrm{d}\mu=\frac{2}{3t^2}\leqslant\frac{2}{3},\lim_{t\to\infty}p(t)=0,$$

$$\int_{t_0}^{+\infty}\int_v^{+\infty}\left[\frac{1}{r(u)}\int_u^{+\infty}\int_c^d q(s,\xi)\mathrm{d}\xi\mathrm{d}s\right]^{\frac{1}{\beta}}\mathrm{d}u\mathrm{d}v=\int_{t_0}^{+\infty}\int_v^{+\infty}\left[\frac{1}{u^2}\int_u^{+\infty}\int_0^1\frac{64\xi}{s^3}\mathrm{d}\xi\mathrm{d}s\right]^{\frac{1}{5}}\mathrm{d}u\mathrm{d}v=+\infty\ ,$$

取 $\rho(t)=t^2,k=2,\eta=0.5$,则

$$Q_1(t)=\left[1-\left(\frac{1}{3}\times 2^{1-\frac{1}{3}}+\frac{(2^{1-\frac{1}{3}}-1)}{2}\right)\frac{2}{3}\right]^7\int_0^1\frac{64\xi}{t^3}\mathrm{d}\xi=\left(\frac{4}{3}-\frac{5}{9}\times 2^{\frac{2}{3}}\right)^7\frac{32}{t^3},Q_2(t)=\frac{1}{t},$$

显然有式(8)成立. 因此由定理1得,方程(26)的所有解振动或收敛于零.

[参考文献]

[1]　HALE J K,LUNEL S V. Introduction to functional differential equations[M]. New York:Springer-Verlag,1977.

[2]　BACULIKOVA B,DZURINA J. Oscillation of third-order neutral differential equations[J]. Mathematical and computer modelling,2010,52(1):215-226.

[3]　YANG L L,XU Z T. Oscillation of certain third-order quasilinear neutral differential equation[J]. Mathematica slovaca,2014,64(1):85-100.

[4]　JIANG Y,JIANG C M,LI T X. Oscillatory behavior of third-order nonliear neutral delay differential equations[J]. Advances in difference equations,2016(1):171-182.

[5]　林文贤. 一类具阻尼项的三阶半线性中立型泛函微分方程的 Philos 型振动结果[J]. 安徽大学学报(自然科学版),2016,55(6):28-32.

[6]　林文贤. 一类具阻尼项的三阶非线性中立型泛函微分方程的振动性[J]. 中山大学学报(自然科学版),2016,40(6):1-4.

[7]　林文贤. 三阶半线性中立型阻尼泛函微分方程的振动性[J]. 华东师范大学学报(自然科学版),2017,2017(3):48-53.

[8]　惠远先,李培峦,戴丽华. 一类三阶非线性分布时滞动力方程的振动结果[J]. 浙江大学学报(理科版),2019,46(3):315-322.

[9]　HARDY G H,LITTLEWOOD J E,POLYA G. Inequalities,second edition[M]. Cambridge:Cambridge University Press,1988.

[10]　KIGURADGE I T,CHANTURIYA T A. Asymptotic properties of solutions of nonautonomous ordinary differential equations[M]. Dordreecht:Kluwer Academic Publishers,1993.

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