doi:10.3969/j.issn.1001-4616.2023.04.001

三角代数上 Jordan 同构的刻画

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[摘要]　设 $U=\mathrm{Tri}(A,M,B)$ 是三角代数,V 是 2-无挠含单位的代数. 本文证明了线性双射 $\varphi:U\to V$ 是 Jordan 同构的充要条件是 φ 保单位且下列条件之一成立:(1)$\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,其中 $x,y\in U$ 满足 $xy=0$. (2)$\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,其中 $x,y\in U$ 满足 $x\circ y=0$. (3)$\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,其中 $x,y\in U$ 满足 $xy=yx=0$.

[关键词]　三角代数,Jordan 同构,零积

[中图分类号]O177.1　[文献标志码]A　[文章编号]1001-4616(2023)04-0001-04

Characterizations of Jordan Isomorphisms of Triangular Algebras

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Abstract:Let $U=\mathrm{Tri}(A,M,B)$ be a triangular algebra and let V be a unitial 2-torsion free algebra. It is shown that a linear bijection $\varphi:U\to V$ is Jordan isomorphism if and only if φ is unital and one of the following statements holds:(1)$\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$ for $x,y\in U$ with $xy=0$;(2)$\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$ for $x,y\in U$ with $x\circ y=0$;(3)$\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$ for $x,y\in U$ with $xy=yx=0$.

Key words:triangular algebra,Jordan isomorphism,zero product

设 A 和 B 是结合代数,$\varphi:A\to B$ 是一个线性双射.若对任意的 $x\in A$ 且 $2x=0$,有 $x=0$,则称 A 是 2-无挠的. 如果对任意 $x,y\in A$,有 $\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,则称 φ 是一个 Jordan 同构,其中 $x\circ y=xy+yx$ 为 Jordan 积. Jordan 同构是算子代数上的一类重要的线性映射,受到了许多学者的关注. 例如:Molnar 等[1]证明了三角矩阵代数 $T_n(F)$ 上的 Jordan 同构是同构或反同构. 这里 F 是至少包含 3 个元素的数域. Beidar 等[2]推广了这个结论,证明了 2-无挠交换环 R 上的上三角矩阵代数上的 Jordan 同构是同构或反同构. Lu[3]证明了套代数上的 Jordan 同构是同构或反同构. Wong[4]研究了上三角矩阵环上的 Jordan 同构,并给出了 Jordan 同构是同构或反同构的条件. 本文将通过由零积、Jordan 零积或交换零积所确定的子集上线性映射的局部性质来刻画三角代数上的 Jordan 同构. 关于线性映射的局部性质的研究主要有两个方面. 一方面是逐点定义的局部映射,如局部导子和局部同构的研究[5-9]. 另一方面是通过线性映射在某些子集上的局部性质来研究它的整体结构性质(见文献[10-16]).

设 A 和 B 是交换环 R 上的含单位元的代数,M 是 (A,B)-忠实双边模. 在通常的矩阵运算下,称

$$\mathrm{Tri}(A,M,B)=\left\{\begin{pmatrix}a & m\\ 0 & b\end{pmatrix}:a\in A,m\in M,b\in B\right\}$$

为三角代数. 设 $U=\mathrm{Tri}(A,M,B)$ 是一个三角代数,1_A 和 1_B 分别是 A 和 B 的单位元. 记

$$\boldsymbol{e}_1=\begin{pmatrix}1_A & 0\\ 0 & 0\end{pmatrix},\ \boldsymbol{e}_2=\begin{pmatrix}0 & 0\\ 0 & 1_B\end{pmatrix},$$

且

收稿日期:2022-02-18.
基金项目:国家自然科学基金项目(11901248).
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$$U_{ij}=\boldsymbol{e}_iU\boldsymbol{e}_j(1\leqslant i\leqslant j\leqslant 2),$$

则 $U=U_{11}+U_{12}+U_{22}$ 且 U_{12} 是 (U_{11},U_{12})-忠实双边模.

1　主要定理及其证明

定理1　设 $U=\mathrm{Tri}(A,M,B)$ 是三角代数,V 是 R 上 2-无挠含单位的代数. 则线性双射 $\varphi:U\to V$ 是 Jordan 同构当且仅当 φ 保单位且下列条件之一成立:

(1) $\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,其中 $x,y\in U$ 满足 $xy=0$.

(2) $\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,其中 $x,y\in U$ 满足 $x\circ y=0$.

(3) $\varphi(x\circ y)=\varphi(x)\circ\varphi(y)$,其中 $x,y\in U$ 满足 $xy=yx=0$.

以下假设 $\varphi:U\to V$ 是一个满足定理1条件(3)的保单位线性双射. 为证明定理1,我们需要以下几个引理.

引理1　对任意幂等元 $p\in U$,有 $\varphi(p)=\varphi(p)^2$.

证明　对任意幂等元 $p\in U$,由 $p(1-p)=(1-p)p=0$ 得

$$0=\varphi(p)\circ\varphi(1-p)=2(\varphi(p)-\varphi(p)^2).$$

从而 $\varphi(p)=\varphi(p)^2$. 证毕.

引理2　设 $u_{ij}\in U_{ij}(1\leqslant i\leqslant j\leqslant 2)$,则

(1) $\varphi(u_{ii})=\varphi(\boldsymbol{e}_i)\varphi(u_{ii})\varphi(\boldsymbol{e}_i)$;

(2) $\varphi(u_{12})=\varphi(\boldsymbol{e}_1)\varphi(u_{12})\varphi(\boldsymbol{e}_2)+\varphi(\boldsymbol{e}_2)\varphi(u_{12})\varphi(\boldsymbol{e}_1)$.

证明　(1) 由于 $u_{11}\boldsymbol{e}_2=\boldsymbol{e}_2u_{11}=0$,则

$$\varphi(u_{11})\varphi(\boldsymbol{e}_2)+\varphi(\boldsymbol{e}_2)\varphi(u_{11})=\varphi(u_{11})\circ\varphi(\boldsymbol{e}_2)=0. \tag{1}$$

从而由引理1和式(1)可得:

$$\varphi(\boldsymbol{e}_2)\varphi(u_{11})\varphi(\boldsymbol{e}_2)=\varphi(\boldsymbol{e}_1)\varphi(u_{11})\varphi(\boldsymbol{e}_2)=\varphi(\boldsymbol{e}_2)\varphi(u_{11})\varphi(\boldsymbol{e}_1)=0.$$

因此,

$$\varphi(u_{11})=\varphi(\boldsymbol{e}_1)\varphi(u_{11})\varphi(\boldsymbol{e}_1)+\varphi(\boldsymbol{e}_1)\varphi(u_{11})\varphi(\boldsymbol{e}_2)+\varphi(\boldsymbol{e}_2)\varphi(u_{11})\varphi(\boldsymbol{e}_1)+\varphi(\boldsymbol{e}_2)\varphi(u_{11})\varphi(\boldsymbol{e}_2)=\varphi(\boldsymbol{e}_1)\varphi(u_{11})\varphi(\boldsymbol{e}_1).$$

类似地,我们有 $\varphi(u_{22})=\varphi(\boldsymbol{e}_2)\varphi(u_{22})\varphi(\boldsymbol{e}_2)$.

(2) 显然,对任意 $v_{12}\in U_{12}$,有

$$\varphi(u_{12})\circ\varphi(u_{12})=0. \tag{2}$$

又 $(\boldsymbol{e}_1-u_{12})(\boldsymbol{e}_2+u_{12})=(\boldsymbol{e}_2+u_{12})(\boldsymbol{e}_1-u_{12})=0$. 则

$$0=\varphi((\boldsymbol{e}_1-u_{12})\circ(\boldsymbol{e}_2+u_{12}))=\varphi(\boldsymbol{e}_1-u_{12})\circ\varphi(\boldsymbol{e}_2+u_{12})=\varphi(\boldsymbol{e}_1)\circ\varphi(u_{12})-\varphi(u_{12})\circ\varphi(\boldsymbol{e}_2)-\varphi(u_{12})\circ\varphi(u_{12}).$$

从而由式(2),

$$\varphi(\boldsymbol{e}_1)\varphi(u_{12})+\varphi(u_{12})\varphi(\boldsymbol{e}_1)-\varphi(u_{12})\varphi(\boldsymbol{e}_2)-\varphi(\boldsymbol{e}_2)\varphi(u_{12})=0.$$

对上式分别左右同乘 $\varphi(\boldsymbol{e}_1)$ 和 $\varphi(\boldsymbol{e}_2)$ 可得:

$$\varphi(\boldsymbol{e}_1)\varphi(u_{12})\varphi(\boldsymbol{e}_1)=\varphi(\boldsymbol{e}_2)\varphi(u_{12})\varphi(\boldsymbol{e}_2)=0.$$

因此,

$$\varphi(u_{12})=\varphi(\boldsymbol{e}_1)\varphi(u_{12})\varphi(\boldsymbol{e}_1)+\varphi(\boldsymbol{e}_1)\varphi(u_{12})\varphi(\boldsymbol{e}_2)+\varphi(\boldsymbol{e}_2)\varphi(u_{12})\varphi(\boldsymbol{e}_1)+\varphi(\boldsymbol{e}_2)\varphi(u_{12})\varphi(\boldsymbol{e}_2)=\varphi(\boldsymbol{e}_1)\varphi(u_{12})\varphi(\boldsymbol{e}_2)+\varphi(\boldsymbol{e}_2)\varphi(u_{12})\varphi(\boldsymbol{e}_1).$$

证毕.

引理3　设 $u_{ij},v_{ij}\in U_{ij}(1\leqslant i\leqslant j\leqslant 2)$,则

(1) $\varphi(u_{11}\circ v_{12})=\varphi(u_{11})\circ\varphi(v_{12})$ 且 $\varphi(u_{12}\circ v_{22})=\varphi(u_{12})\circ\varphi(v_{22})$;

(2) $\varphi(u_{11}\circ v_{11})=\varphi(u_{11})\circ\varphi(v_{11})$ 且 $\varphi(u_{22}\circ v_{22})=\varphi(u_{22})\circ\varphi(v_{22})$.

证明　(1) 由于 $(u_{11}-u_{11}v_{12})(v_{12}+\boldsymbol{e}_2)=(v_{12}+\boldsymbol{e}_2)(u_{11}-u_{11}v_{12})=0$. 则由引理2和式(2)可得:

$$0=\varphi(u_{11}-u_{11}v_{12})\circ\varphi(v_{12}+\boldsymbol{e}_2)=\varphi(u_{11})\circ\varphi(v_{12})-\varphi(u_{11}v_{12})\circ\varphi(\boldsymbol{e}_2)+\varphi(u_{11})\circ\varphi(\boldsymbol{e}_2)-\varphi(u_{11}v_{12})\circ\varphi(v_{12})=\varphi(u_{11})\circ\varphi(v_{12})-\varphi(u_{11}v_{12}).$$

这说明

$$\varphi(u_{11}\circ v_{12})=\varphi(u_{11})\circ\varphi(v_{12}).\tag{3}$$

类似地,由$(u_{12}+\boldsymbol{e}_1)(v_{22}-u_{12}v_{22})=(v_{22}-u_{12}v_{22})(u_{12}+\boldsymbol{e}_1)=0$,我们有

$$\varphi(u_{12}\circ v_{22})=\varphi(u_{12})\circ\varphi(v_{22}).\tag{4}$$

(2)由式(3),一方面

$$\varphi((u_{11}\circ v_{11})\circ v_{12})=\varphi(u_{11}\circ v_{11})\circ\varphi(v_{12}),\tag{5}$$

另一方面,我们有

$$\begin{gathered}\varphi((u_{11}\circ v_{11})\circ v_{12})=\varphi(u_{11}\circ(v_{11}v_{12}))+\varphi(v_{11}\circ(u_{11}v_{12}))=\varphi(u_{11})\circ\varphi(v_{11}v_{12})+\varphi(v_{11})\circ\varphi(u_{11}v_{12})=\\ \varphi(u_{11})\circ(\varphi(v_{11})\circ\varphi(v_{12}))+\varphi(v_{11})\circ(\varphi(u_{11})\circ\varphi(v_{12}))=(\varphi(u_{11})\circ\varphi(v_{11}))\circ\varphi(v_{12})+\\ 2\varphi(u_{11})\varphi(v_{12})\varphi(v_{11})+2\varphi(v_{11})\varphi(v_{12})\varphi(u_{11}).\end{gathered}$$

由引理 3 知,$\varphi(u_{11})\varphi(v_{12})\varphi(v_{11})=\varphi(v_{11})\varphi(v_{12})\varphi(u_{11})=0$. 从而由上式,

$$\varphi((u_{11}\circ v_{11})\circ v_{12})=(\varphi(u_{11})\circ\varphi(v_{11}))\circ\varphi(v_{12}).\tag{6}$$

由式(5)和式(6),则对任意 $v_{12}\in U_{12}$,有

$$(\varphi(u_{11}\circ v_{11})-\varphi(u_{11})\circ\varphi(v_{11}))\circ\varphi(v_{12})=0.\tag{7}$$

由于 φ 是满射,则存在 $x\in U$ 使得

$$\varphi(x)=\varphi(u_{11}\circ v_{11})-\varphi(u_{11})\circ\varphi(v_{11}).\tag{8}$$

从而由引理 2 可知,

$$\begin{gathered}\varphi(x)=\varphi(\boldsymbol{e}_1)\varphi(x)\varphi(\boldsymbol{e}_1)=\varphi(\boldsymbol{e}_1)\varphi(\boldsymbol{e}_1x\boldsymbol{e}_1)\varphi(\boldsymbol{e}_1)+\varphi(\boldsymbol{e}_1)\varphi(\boldsymbol{e}_1x\boldsymbol{e}_2)\varphi(\boldsymbol{e}_1)+\\ \varphi(\boldsymbol{e}_1)\varphi(\boldsymbol{e}_2x\boldsymbol{e}_2)\varphi(\boldsymbol{e}_1)=\varphi(\boldsymbol{e}_1)\varphi(\boldsymbol{e}_1x\boldsymbol{e}_1)\varphi(\boldsymbol{e}_1)=\varphi(x_{11}),\end{gathered}$$

其中 $x_{11}=\boldsymbol{e}_1x_1\boldsymbol{e}_1\in U_{11}$,于是由式(3)和式(7),

$$\varphi(x_{11}v_{12})=\varphi(x_{11}\circ v_{12})=\varphi(x_{11})\circ\varphi(v_{12})=\varphi(x)\circ\varphi(v_{12})=0.$$

由于 φ 是单射,则 $x_{11}U_{12}=\{0\}$,并由 U_{12}的忠实性得 $x_{11}=0$. 从而 $\varphi(x)=\varphi(x_{11})=0$,并由式(8)得

$$\varphi(u_{11}\circ v_{11})=\varphi(u_{11})\circ\varphi(v_{11}).$$

类似地,我们由式(4)可得 $\varphi(u_{22}\circ v_{22})=\varphi(u_{22})\circ\varphi(v_{22})$. 证毕.

定理 1 的证明　由文献[4]的结论知,如果 φ 是 Jordan 同构,则 φ 保单位且条件(1),(2)和(3)之一成立. 反过来,容易验证(1)⇒(3)和(2)⇒(3)分别成立. 以下我们假设 φ 保单位且条件(3)成立.

设 $u,v\in U$,则存在 $u_{ij},v_{ij}\in U_{ij}$使得

$$u=u_{11}+u_{12}+u_{22},v=v_{11}+v_{12}+v_{22}.$$

从而由引理 3 及 $\varphi(u_{ii})\circ\varphi(v_{jj})=\varphi(u_{12})\circ\varphi(v_{12})=0(i\neq j)$可知,

$$\begin{gathered}\varphi(u\circ v)=\varphi((u_{11}+u_{12}+u_{22})\circ(v_{11}+v_{12}+v_{22}))=\varphi(u_{11}\circ v_{11})+\varphi(u_{11}\circ v_{12})+\varphi(u_{12}\circ v_{11})+\varphi(u_{12}\circ v_{22})+\varphi(u_{22}\circ v_{12})+\\ \varphi(u_{22}\circ v_{22})=\varphi(u_{11})\circ\varphi(v_{11})+\varphi(u_{11})\circ\varphi(v_{12})+\varphi(u_{12})\circ\varphi(v_{11})+\varphi(u_{12})\circ\varphi(v_{22})+\varphi(u_{22})\circ\varphi(v_{12})+\\ \varphi(u_{22})\circ\varphi(v_{22})=(\varphi(u_{11})+\varphi(u_{12})+\varphi(u_{22}))\circ(\varphi(v_{11})+\varphi(v_{12})+\varphi(v_{22}))=\varphi(u)\circ\varphi(v).\end{gathered}$$

因此,φ 是 Jordan 同构. 证毕.

设 R 是含单位的交换环,$\boldsymbol{M}_{n\times k}(R)$表示 R 上全体 $n\times k$ 矩阵. 对 $n\geqslant 2$ 且 $1\leqslant m\leqslant n$,$\boldsymbol{M}_n(R)$中形如

$$\begin{pmatrix}\boldsymbol{M}_{k_1}(R) & \boldsymbol{M}_{k_1\times k_2}(R) & \cdots & \boldsymbol{M}_{k_1\times k_m}(R)\\ 0 & \boldsymbol{M}_{k_2}(R) & \cdots & \boldsymbol{M}_{k_2\times k_m}(R)\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{M}_{k_m}(R)\end{pmatrix}$$

的子代数记作 $T_n^{\bar{k}}(R)$,并称为上三角矩阵块代数,其中 $\bar{k}=(k_1,\cdots,k_m)$且 $k_1+\cdots+k_m=n$.

设 $B(H)$表示复 Hilbert 空间 H 上全体有界线性算子构成的代数. H 上的套 N 是 $B(H)$中包含 0 和 I 且在强算子拓扑下闭的全序投影族,称

$$\operatorname{Alg} N=\{T\in B(H):PTP=TP,P\in N\}$$

为关于 N 的套代数. 我们知道上三角矩阵代数与非平凡套代数均为三角代数,从而由[4]和[3]的结论以及定理 1,我们有下列推论.

推论 1　设 R 是含单位元的 2-无挠交换环. S_n 是 $\boldsymbol{M}_n(R)$中含单位的子代数,如果 $\varphi:T_n^{\bar{k}}(R)\to S_n$ 是一

个保单位的线性双射且下列条件之一成立:

(1) $\varphi(A\circ B)=\varphi(A)\circ\varphi(B)$,其中 $A,B\in T_n^{\bar{k}}(R)$ 满足 $AB=0$,

(2) $\varphi(A\circ B)=\varphi(A)\circ\varphi(B)$,其中 $A,B\in T_n^{\bar{k}}(R)$ 满足 $A\circ B=0$,

(3) $\varphi(A\circ B)=\varphi(A)\circ\varphi(B)$,其中 $A,B\in T_n^{\bar{k}}(R)$ 满足 $AB=BA=0$,

则 φ 要么是一个同构,要么是一个反同构.

推论 2　设 N 和 M 是复 Hilbert 空间 H 上的非平凡套,φ:Alg $N\to$ Alg M 是一个线性双射且 $\varphi(I)=I$,如果下列条件之一成立:

(1) $\varphi(A\circ B)=\varphi(A)\circ\varphi(B)$,其中 $A,B\in$ Alg N 满足 $AB=0$,

(2) $\varphi(A\circ B)=\varphi(A)\circ\varphi(B)$,其中 $A,B\in$ Alg N 满足 $A\circ B=0$,

(3) $\varphi(A\circ B)=\varphi(A)\circ\varphi(B)$,其中 $A,B\in$ Alg N 满足 $AB=BA=0$,

则存在可逆算子 $T\in B(H)$,使得对任意 $A\in$ Alg N 有 $\varphi(A)=TAT^{-1}$ 或 $\varphi(A)=TJA^*JT^{-1}$,其中 J 是共轭线性对合.

注 1　推论 2 在更弱的条件下得到了与文[14]相同的结论. 文[17]证明了保单位且双边保 Jordan 零积的可加满射是双射,并由此给出了套代数上此类映射的具体结构,而推论 2 给出了套代数上保单位且单边保 Jordan 零积的线性双射的具体结构.

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